A Scientific Study: k-Dimensional Tic-Tac-Toe

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A typical game of Tic-Tac-Toe is played by two people. Each take turn to mark in a three-by-three, two-dimensional grid. The objective of the game is to place three marks of the same kind in a row either in a horizontal, vertical, or diagonal direction. The following picture from Wikipedia demonstrates one of the possible sequences leading to a winning condition in the horizontal direction.

We will expand the popular definition of Tic-Tac-Toe to k-Tic-Tac-Toe, a game of two or more players taking turn to mark in a k-dimensional space, each dimension with a set of domain $\mathbf{Z}_3 = \{0, 1, 2\}$. Notice we can apply our experience with $\mathbf{Z}_2 = \{0, 1\}$. If we define vector addition and scalar multiplication carefully, which we will discuss later, we will have a vector space $\mathbf{Z}_3^k$. The number of grids in a k-dimensional game is 3^k. A player wins a game if three marks of the same kind is connected in a line.

0-Tic-Tac-Toe and 1-Tic-Tac-Toe

Since $3^0 = 1$, 0-Tic-Tac-Toe has to be played within one grid, which is not possible. Similarly, $3^1 = 3$, a player cannot satisfy the winning condition without violating the rules in a 1-Tic-Tac-Toe.

2-Tic-Tac-Toe

This is the original two-dimensional Tic-Tac-Toe. We start our analysis by introducing position vectors that describes the location of a mark, vector addition and scalar multiplication rules in this vector space, and the conditions under which these vectors must meet in order to qualify as a wining set.

The tables to the left illustrate the addition and scalar multiplication rules to maintain the vector space property. These rules are analogous to those of the bit matrices.

·	*0*	*1*	*2*
0	0	0	0
1	0	1	2
2	0	2	1

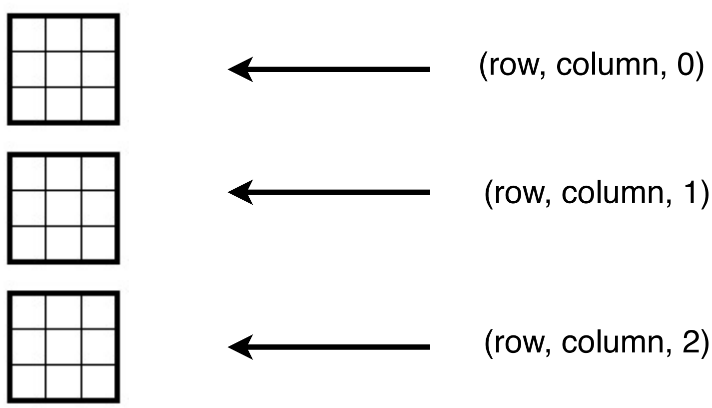
+	*0*	*1*	*2*
0	0	1	2
1	1	2	0
2	2	0	1

(0, 0)	(0, 1)	(0, 2)
(1, 0)	(1, 1)	(1, 2)
(2, 0)	(2, 1)	(2, 2)

The coordinate system in a 2-Tic-Tac-Toe is denoted by the (row, column) pair. The winning position vectors must lie on the same line. In other words, *a* set of position vectors locating the same kind of mark **W** = {W_1, W_2, W_3} wins the game if and only if **W** is a subset of **S** = {I + kD}, where I is an element in **W**, D is the difference vector between two distinct element in **W**, and k is a real number.

This general definition works for all the k-dimensional Tic-Tac-Toe's. For example, in the two-dimensional case, the set {(2, 0), (1, 1), (0, 2)} wins the game because each element is in the set {(1, 1) + k[(2, 0) - (1, 1)]}. The choice of position vectors is arbitrary. The set {(0, 2) + k[(1, 1)-(0, 2)]} and the set {(2, 0) +k[(2, 0)-(0, 2)]} contains the solution as well.

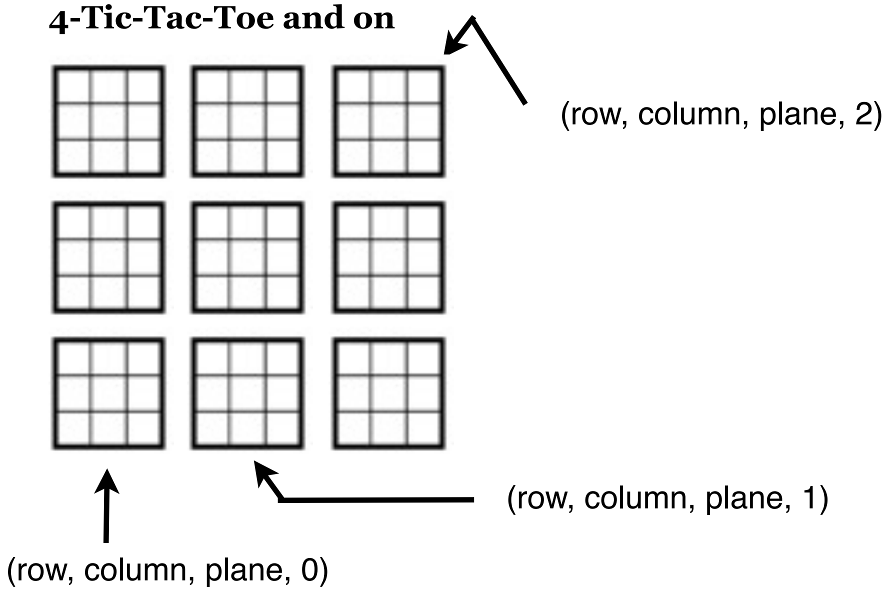
3-Tic-Tac-Toe



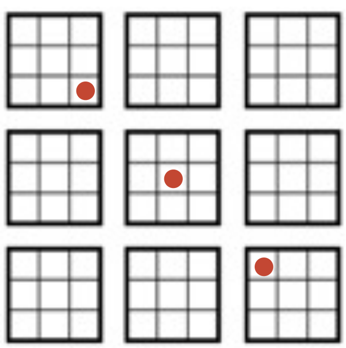
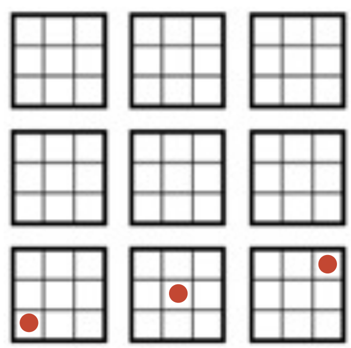
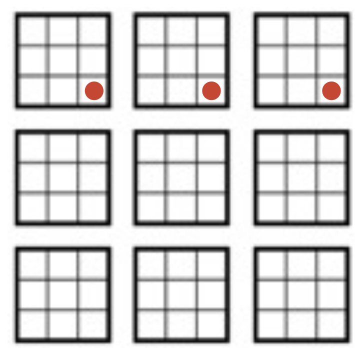
In three-dimensional Tic-Tac-Toe, which has 3^3 = 27 grids, we use a third component to indicate which *plane* a given mark is. Using the same winning condition, which we will restate here: *A* set of position vectors locating the same kind of mark **W** = {W_1, W_2, W_3} wins the game if and only if **W** is a subset of **S** = {I + kD}, where I is an element in **W**, D is the difference, the 'slope', vector between two distinct element in **W**, and k is a real number, the set {(0, 0, 0), (1, 1, 1), (2, 2, 2)} and the set {(0, 1, 0), (1, 1, 1), (2, 1, 2)} both satisfies the condition for winning. One way to visualize this is to imagine the three planes are stacked together to form a cube, and intuitively, any straight lines in the diagonal, horizontal, and vertical direction of the cube contains a winning set.

Notice the solutions from a 2-Tic-Tac-Toe game is valid in 3-Tic-Tac-Toe space. When n > k, the solution from a k-Tic-Tac-Toe game works for any n-Tic-Tac-Toe game; the elements of the 'lower dimension' winning set in the n-Tic-Tac-Toe game will just have the same numbers from the k+1st components on.

4-Tic-Tac-Toe and on



A fourth component of the position vector is introduced to describe which *cube* a mark is. Though we are now entering a four-dimensional game, the same condition for winning the game still applies. Examples of winning sets include {(0, 0, 2, 2), (1, 1, 1, 1), (2, 2, 0, 0)}, {(0, 2, 2, 2), (1, 1, 2, 1), (2, 0, 2, 0)}, and {(2, 2, 1, 0), (2, 2, 1, 1), (2, 2, 1, 2)} as the following illustrations show respectively.

In conclusion, this condition to win continues for higher-dimensional Tic-Tac-Toe games as long as the three marks of the same kind form a valid subset in the set of winning 'lines.'

References

Dr. George's Science Web
Tic-tac-toe @ Wikipedia