Probabilistic Analysis
Solutions to Exercises

1. Steps 3-9 take $ck\log_2(k)$ time.

   (a) In the worst case, $k = n$, so $T(n) \in \Theta(n\log_2(n))$.

   (b) $\text{Prob}(k = j) = 1/n$ for $1 \le j \le n$.

$$\begin{aligned}
ET(n) &= \sum_{j=1}^{n} \text{Prob}(k=j)\text{Time}(k=j) = \sum_{j=1}^{n}(1/n)ck\log_2(k) = (1/n)\sum_{j=1}^{n} ck\log_2(k) \\
&\le (c/n)\sum_{j=1}^{n} n\log_2(n) = (c/n)n(n\log_2(n)) = cn\log_2(n) \in O(n\log_2(n)). \\
ET(n) &= (1/n)\sum_{j=1}^{n} ck\log_2(k) \ge (c/n)\sum_{j=n/2}^{n} k\log_2(k) \ge (c/n)\sum_{j=n/2}^{n}(n/2)\log_2(n/2) \\
&= (c/n)(n/2)(n/2)(\log_2(n)-1) = cn\log_2(n)/4 - cn/4 \in \Omega(n\log_2(n)).
\end{aligned}$$

   Thus, $ET(n) \in \Theta(n\log_2(n))$.

2. Steps 4-10 take $cn\log_2(n)$ time.

   (a) In the worst case, $k < \log_2(n)$, so $T(n) \in \Theta(n\log_2(n))$.

   (b) $\text{Prob}(k < \log_2(n)) = \lfloor \log_2(n) \rfloor / n \approx \log_2(n)/n$.

$$\begin{aligned}
ET(n) &= \text{Prob}(k < \log_2(n))\text{Time}(k < \log_2(n)) + \text{Prob}(k \ge \log_2(n))\text{Time}(k \ge \log_2(n)) \\
&= \frac{\log_2(n)}{n}(cn\log_2(n)) + \left(1 - \frac{1}{\log_2(n)}\right)c \approx (c(\log_2(n))^2 + c) \in \Theta((\log_2(n))^2).
\end{aligned}$$

   So $ET(n) \in \Theta((\log_2(n))^2)$.

3. Steps 2-4 take $c\sqrt{n}$ time.

   (a) In the worst case, k is less than $2n/3$.

$$\begin{aligned}
T(n) &= c\sqrt{n} + T(n-5) = c\sqrt{n} + c\sqrt{n-5} + c\sqrt{n-10} + \ldots + c \\
&\le \underbrace{c\sqrt{n} + c\sqrt{n} + \ldots + c\sqrt{n}}_{n/5} = c\sqrt{n}(n/5) \in O(n^{1.5}). \\
T(n) &= c\sqrt{n} + c\sqrt{n-5} + c\sqrt{n-10} + \ldots + c \\
&\ge c\sqrt{n} + c\sqrt{n-5} + c\sqrt{n-10} + \ldots + c\sqrt{n/2} \ge \underbrace{c\sqrt{n/2} + c\sqrt{n/2} + \ldots + c\sqrt{n/2}}_{n/10} \\
&= c\sqrt{n/2}(n/10) = cn^{1.5}/(10\sqrt{2}) \in \Omega(n^{1.5}).
\end{aligned}$$

   So $T(n) \in \Theta(n^{1.5})$.

   (b) $\text{Prob}(k < 2n/3) = (2n/3)/n = 2/3$.

$$\begin{aligned}
ET(n) &= \text{Prob}(k < 2n/3)ET(k < 2n/3) + \text{Prob}(k \ge 2n/3)ET(k < 2n/3) \\
&= (2/3)(c\sqrt{n} + ET(n-5)) + (1/3)c\sqrt{n} = c\sqrt{n} + (2/3)ET(n-5) \ge c\sqrt{n}. \\
ET(n) &= c\sqrt{n} + (2/3)ET(n-5) \\
&= c\sqrt{n} + (2/3)c\sqrt{n-5} + (2/3)^2c\sqrt{n-10} + (2/3)^3c\sqrt{n-15} + \ldots + (2/3)^{n/5}c \\
&\le c\sqrt{n} + (2/3)c\sqrt{n} + (2/3)^2c\sqrt{n} + (2/3)^3c\sqrt{n} + \ldots + (2/3)^{n/5}c\sqrt{n} \\
&\le c\sqrt{n}(1 + 2/3 + (2/3)^2 + (2/3)^3 + \ldots) = c\sqrt{n}\frac{1}{1-(2/3)} = 3c\sqrt{n}.
\end{aligned}$$

   Thus $c\sqrt{n} \le ET(n) \le 3c\sqrt{n}$ and $ET(n) \in \Theta(\sqrt{n})$.

4. Steps 2-4 take c time.

   (a) In the worst case $c_1 = c_2$ so statement 6 is executed.

$$\begin{aligned} T(n) &= c + T(n-4) + T(n-7) \geq T(n-4) + T(n-7) \geq 2T(n-7) \\ &\geq 2 * 2T(n-14) \geq 2 * 2 * 2T(n-21) \geq \underbrace{2 * 2 * 2 * \ldots * 2}_{n/7} * T(1) \\ &\geq 2^{n/7} c \in \Omega(2^{n/7}). \end{aligned}$$

   Since $T(n) \in \Omega(2^{n/7})$, $T(n)$ has an exponential lower bound.

   (b) Prob$(c_1 = c_2)$ is 1/2.

$$\begin{aligned} ET(n) &= \text{Prob}(c_1 = c_2) ET(c_1 = c_2) + \text{Prob}(c_1 \neq c_2) ET(c_1 \neq c_2) \\ &= (1/2)(c + T(n-4) + T(n-7)) + (1/2)c \\ &= c + (1/2)T(n-4) + (1/2)T(n-7). \\ ET(n) &= c + (1/2)T(n-4) + (1/2)T(n-7) \\ &\leq c + (1/2)T(n-4) + (1/2)T(n-4) = c + T(n-4) = c + c + T(n-8) \\ &= \underbrace{c + c + \ldots + T(1)}_{n/4} = \underbrace{c + c + \ldots + c}_{n/4} = cn/4. \\ ET(n) &= c + (1/2)T(n-4) + (1/2)T(n-7) \\ &\geq c + (1/2)T(n-7) + (1/2)T(n-7) = c + T(n-7) = c + c + T(n-14) \\ &= \underbrace{c + c + \ldots + T(1)}_{n/7} = \underbrace{c + c + \ldots + c}_{n/7} = cn/7. \end{aligned}$$

   Since $cn/7 \leq ET(n) \leq cn/4$, $ET(n) \in \Theta(n)$.

5. Steps 1-6 take cn time.

   (a) In the worst case c is always heads.

$$\begin{aligned} T(n) &= cn + 4T(n/4) = cn + 4(cn/4 + 4T(n/4^2)) = cn + cn + 4^2 T(n/4^2) \\ &= \underbrace{cn + cn + cn + \ldots + cn + 4^{\log_4(n)} T(1)}_{\log_4(n)} \\ &= \underbrace{cn + cn + cn + \ldots + cn + cn}_{\log_4(n)} = cn \log_4(n). \end{aligned}$$

   Thus, $T(n) \in \Theta(n \log_2(n))$.

   (b)

$$\begin{aligned} ET(n) &= cn + 4 * ET(\text{steps } 8-11) = cn + 4((1/2)ET(n/4)) = cn + 2ET(n/4) \\ &= cn + 2(cn/4 + 2ET(n/4^2)) = cn + cn/2 + 2^2 ET(n/4^2) \\ &= \underbrace{cn + cn/2 + cn/4 + cn/8 + \ldots + 2^{\log_4(n)} ET(1)}_{\log_4(n)} \\ &= \underbrace{cn + cn/2 + cn/4 + cn/8 + \ldots + c\sqrt{n}}_{\log_4(n)} \\ &= \underbrace{cn + cn/2 + cn/4 + cn/8 + \ldots + cn/\sqrt{n}}_{\log_4(n)} \\ &\leq cn + cn/2 + cn/4 + cn/8 + \ldots \leq cn(1 + 1/2 + 1/4 + 1/8 + \ldots) \\ &\leq cn \frac{1}{1 - (1/2)} = 2cn. \\ ET(n) &= cn + 2ET(n/4) \geq cn. \end{aligned}$$

   Since $cn \leq ET(n) \leq 2cn$, $ET(n) \in \Theta(n)$.