Asymptotic Notation

Asymptotic Running Time

What is the asymptotic running time of the following code fragment?

```
for i ← 1 to 3n^2 + 5n + 8 do
    for j ← 1 to √(7n + 6) do
        x ← x + i − j;
    end
end
```

Asymptotic Notation

$f(n) \in O(n^2)$ if there exists $c, n_0 > 0$ such that:

$$f(n) \leq cn^2 \quad \text{for all } n \geq n_0.$$

$f(n) \in \Omega(n^2)$ if there exists $c, n_0 > 0$ such that:

$$f(n) \geq cn^2 \quad \text{for all } n \geq n_0.$$

$f(n) \in \Theta(n^2)$ if there exists $c_1, c_2, n_0 > 0$ such that:

$$c_1 n^2 \leq f(n) \leq c_2 n^2 \quad \text{for all } n \geq n_0.$$

Asymptotic Notation

"$f(n) \in O(g(n))$" means:

$f(n)$ grows at most as fast as $g(n)$.

"$f(n) \in \Omega(g(n))$" means:

$f(n)$ grows at least as fast as $g(n)$.

"$f(n) \in \Theta(g(n))$" means:

$f(n)$ grows at at the same rate as $g(n)$.

Asymptotic Notation

$f(n) \in O(g(n))$ if there exists $c, n_0 > 0$ such that:

$$f(n) \le cg(n) \quad \text{for all } n \ge n_0.$$

$f(n) \in \Omega(g(n))$ if there exists $c, n_0 > 0$ such that:

$$f(n) \ge cg(n) \quad \text{for all } n \ge n_0.$$

$f(n) \in \Theta(g(n))$ if there exists $c_1, c_2, n_0 > 0$ such that:

$$c_1 g(n) \le f(n) \le c_2 g(n) \quad \text{for all } n \ge n_0.$$

Asymptotic Notation: Examples

- $5n^2 + 6n + 8 \in O(n^3)$;
- $2^n \in \Omega(n^2)$;
- $\sqrt{6n^3 + 7n^2 + 3n + 5} \in \Theta(n^{1.5})$;

Asymptotic Notation: Example

Claim: $\sqrt{6n^3 + 7n^2 + 3n + 5} \in \Theta(n^{1.5})$.

Proof:

$$\sqrt{6n^3 + 7n^2 + 3n + 5} \geq \sqrt{6n^3} = \sqrt{6}n^{1.5}.$$

$$\begin{aligned} \sqrt{6n^3 + 7n^2 + 3n + 5} &\leq \sqrt{6n^3 + 7n^3 + 3n^3 + 5n^3} \text{ for } n \geq 1 \\ &\leq \sqrt{21n^3} = \sqrt{21}n^{1.5}. \end{aligned}$$

Thus, $\sqrt{6}n^{1.5} \leq \sqrt{6n^3 + 7n^2 + 3n + 5} \leq \sqrt{21}n^{1.5}$,
and $\sqrt{6n^3 + 7n^2 + 3n + 5} \in \Theta(n^{1.5})$.

Asymptotic Notation

$f(n) \in O(g(n))$ if there exists $c, n_0 > 0$ such that:

$$f(n) \leq cg(n) \quad \text{for all } n \geq n_0.$$

$f(n) \in \Omega(g(n))$ if there exists $c, n_0 > 0$ such that:

$$f(n) \geq cg(n) \quad \text{for all } n \geq n_0.$$

$f(n) \in \Theta(g(n))$ if there exists $c_1, c_2, n_0 > 0$ such that:

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \text{for all } n \geq n_0.$$

Asymptotic Notation

$f(n) \in O(g(n))$ if there exists $c > 0$ such that:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} \leq c.$$

$f(n) \in \Omega(g(n))$ if there exists $c > 0$ such that:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} \geq c.$$

$f(n) \in \Theta(g(n))$ if there exists $c_1, c_2 > 0$ such that:

$$c_1 \leq \lim_{n \to \infty} \frac{f(n)}{g(n)} \leq c_2.$$

Asymptotic Notation

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then

$$f(n) \in O(g(n)) \text{ but } f(n) \notin \Theta(g(n)).$$

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then

$$f(n) \in \Omega(g(n)) \text{ but } f(n) \notin \Theta(g(n)).$$

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0$ $(c \neq \infty)$, then

$$f(n) \in \Theta(g(n)).$$

Asymptotic Notation: Examples

Compare:

- $2n^4 + 4n^3 + n^2$ and $9n^3 + 7n^2 + 6n$;
- $n^{1/2}$ and $n^{1/4}$;
- $\log_2(n)$ and $\log_3(n)$;
- $\log_2(n)$ and $\log_2(n^2)$;
- $\log_2(n)$ and $(\log_2(n))^2$);
- $\log_2(n)$ and $n^{0.1}$;
- n^3 and 3^n;
- n and $n \log_2(n)$;
- n^2 and $n \log_2(n)$;
- n and $\log_2(3^n)$.

Math Equalities

Logarithms:

$$\log_a(n) = \frac{\log_2(n)}{\log_2(a)}.$$

l'Hopital's rule: If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Example of using limits

Compare $f(n) = \log_3(2n^3 + 5)$ and $g(n) = \log_4(3n^2 + 6n)$ using limits.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log_3(2n^3+5)}{\log_4(3n^2+6n)} &\leq \lim_{n\to\infty} \frac{\log_3(2n^3+5n^3)}{\log_4(3n^2)} = \lim_{n\to\infty} \frac{\log_3(7n^3)}{\log_4(3n^2)} \\
&= \lim_{n\to\infty} \frac{\log_4(7n^3)/\log_4(3)}{\log_4(3n^2)} = \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{\log_4(7n^3)}{\log_4(3n^2)} \\
&= \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{3\log_4(n) + \log_4(7)}{2\log_4(n) + \log_4(3)} \\
&= \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{(1/\log_4(n))}{(1/\log_4(n))} \frac{(3\log_4(n) + \log_4(7))}{(2\log_4(n) + \log_4(3))} \\
&= \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{3 + \log_4(7)/\log_4(n)}{2 + \log_4(3)/\log_4(n)} \\
&= \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{3}{2} = \frac{3}{2\log_4(3)}.
\end{aligned}$$

Example of using limits (continued)

Compare $f(n) = \log_3(2n^3 + 5)$ and $g(n) = \log_4(3n^2 + 6n)$ using limits.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log_3(2n^3+5)}{\log_4(3n^2+6n)} &\geq \lim_{n\to\infty} \frac{\log_3(2n^3)}{\log_4(3n^2+6n^2)} = \lim_{n\to\infty} \frac{\log_3(2n^3)}{\log_4(9n^2)} \\
&= \lim_{n\to\infty} \frac{\log_4(n^3)/\log_4(3)}{\log_4(9n^2)} = \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{\log_4(2n^3)}{\log_4(9n^2)} \\
&= \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{3\log_4(n) + \log_4(2)}{2\log_4(n) + \log_4(9)} \\
&= \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{(1/\log_4(n))}{(1/\log_4(n))} \frac{3\log_4(n) + \log_4(2)}{2\log_4(n) + \log_4(9)} \\
&= \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{3 + \log_4(2)/\log_4(n)}{2 + \log_4(9)/\log_4(n)} = \frac{1}{\log_4(3)} \lim_{n\to\infty} \frac{3}{2} = \frac{3}{2\log_4(3)}.
\end{aligned}$$

Since $\lim_{n\to\infty} f(n)/g(n) = \frac{3}{2\log_4(3)}$, $f(n) \in \Theta(g(n))$.

Asymptotic Notation

- $f(n) \in O(g(n))$ if and only if $g(n) \in \Omega(f(n))$.
- $f(n) \in \Theta(g(n))$ if and only if
$$f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n)).$$
- Some (older) books use the (bad) notation "$f(n) = O(g(n))$" in place of "$f(n) \in O(g(n))$".
- Text uses $\overset{\infty}{\forall} n$ in place of "for all $n \geq n_0$".

The Hierarchy

- $\Theta(n^n)$
- $\Theta(3^n)$
- $\Theta(2^n)$
- $\Theta(n^3)$
- $\Theta(n^2)$
- $\Theta(n \log(n))$
- $\Theta(n)$
- $\Theta(n^{0.5})$
- $\Theta(n^{0.1})$
- $\Theta((\log(n))^2)$
- $\Theta(\log(n))$
- $\Theta(1)$

Sample "for" loop

```
function func(n)
x ← 0;
for i ← 1 to n do
    for j ← 1 to i do
        x ← x + (i − j);
    end
end
return (x);
```

Sample "for" loop

function `func`(n)

if $(n > 100000)$ **then return** (0);

$x \leftarrow 0$;

for $i \leftarrow 1$ **to** n **do**

    for $j \leftarrow 1$ **to** n **do**

        $x \leftarrow x + (i - j)$;

    end

end

return (x);

Sample "for" loop

```
function func(n)
if (n < 100000) then return (0);
x ← 0;
for i ← 1 to n do
    for j ← 1 to n do
        x ← x + (i − j);
    end
end
return (x);
```

Analysis of Algorithms

function ContainsDuplicate

Input : Array A of n elements.

```
function ContainsDuplicate(A[ ],n)
for i ← 2 to n do
    for j ← 1 to i − 1 do
        if (A[i] = A[j]) then
            return (true);
        end
    end
end
return (false);
```

Sample for loops

function `func`(n)

```
x ← 0;
for i ← 1 to n do
    for j ← 1 to ⌊√i⌋ do
        x ← x + (i − j);
    end
end
return (x);
```

Sample for loops

function `func`(n)

$x \leftarrow 0$;

for $i \leftarrow 1$ **to** $\lfloor \sqrt{n} \rfloor$ **do**

    for $j \leftarrow 1$ **to** $\lfloor \sqrt{n} \rfloor - i$ **do**

        $x \leftarrow x + (i - j)$;

    end

end

return (x);

Sample while loops

```
function func(n)
x ← 0;
i ← 7;
while (i ≤ n) do
    x ← x + i;
    i ← i + 3;
end
return (x);
```

Sample while loops

```
function func(n)
x ← 0;
i ← 1;
while (i ≤ n) do
    x ← x + i;
    i ← 2 * i ;                    /* Note: Multiplication */
end
return (x);
```

Sample while loops

function `func`(n)

$x \leftarrow 0$;

$i \leftarrow 7$;

while $(i \leq n)$ **do**

  $x \leftarrow x + i$;

  $i \leftarrow 2 * i$;

end

return (x);

Sample while loops

```
function func(n)
x ← 0;
i ← 1;
while (i ≤ n) do
    x ← x + i;
    i ← 3 * i;
end
return (x);
```

Sample loops

```
function func(n)
x ← 0;
for i ← 1 to n do
    j ← 1;
    while (j ≤ n) do
        x ← x + (i − j);
        j ← 2 ∗ j;
    end
end
return (x);
```

Sample loops

```
function func(n)
x ← 0;
for i ← 1 to n do
    j ← 1;
    while (j ≤ i) do
        x ← x + (i − j);
        j ← 2 ∗ j;
    end
end
return (x);
```

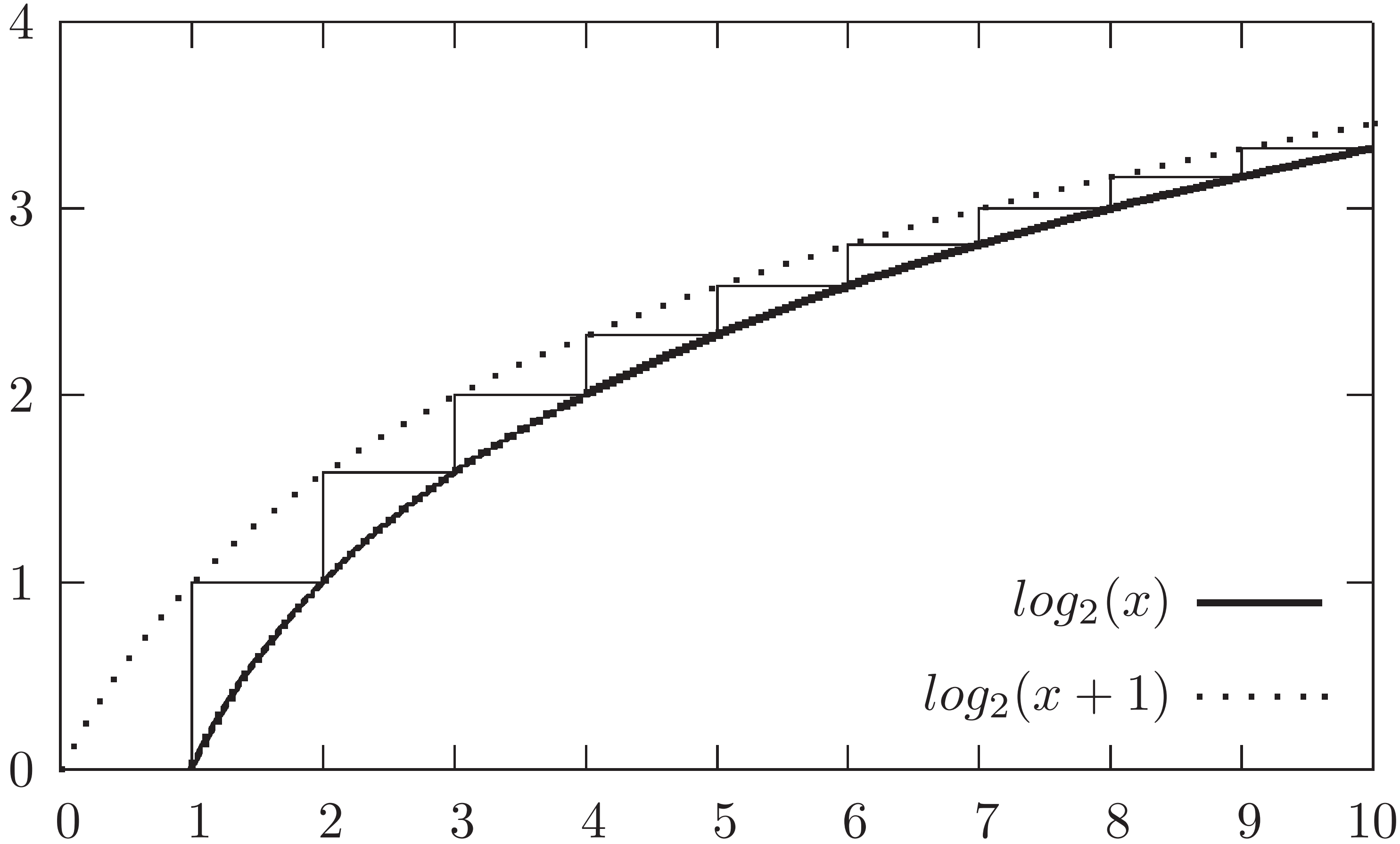
Evaluating $\sum f(i)$

To evaluate $\sum_{i=1}^{n} f(i)$:

- Compute upper and lower bounds on $\sum_{i=1}^{n} f(i)$.
- Use integrals (if possible):

$$\sum_{i=1}^{n} f(i) \approx \int_{1}^{n} f(x)dx.$$

Bounding a sum using integrals



Nested for loops

function `func`(n)

$x \leftarrow 0$;

for $i \leftarrow 1$ **to** n **do**

  for $j \leftarrow 1$ **to** i **do**

    for $k \leftarrow j$ **to** i **do**

      $x \leftarrow x + (k * i - j)$;

    end

  end

end

return (x);

Nested loops

```
function func(n)
x ← 0;
i ← 1;
while (i ≤ n) do
    for j ← 1 to i do
        x ← x + (i − j);
    end
    i ← 2 ∗ i;
end
```

Nested loops

function `func`(n)

$x \leftarrow 0$;
$i \leftarrow 1$;
while $(i \leq n)$ **do**
    for $j \leftarrow 1$ **to** i **do**
        $x \leftarrow x + (i - j)$;
    end
    $i \leftarrow 3 * i$;
end

Nested loops

```
function func(n)
x ← 0;
i ← 1;
while (i ≤ n) do
    for j ← 1 to ⌊n/i⌋ do
        x ← x + (i − j);
    end
    i ← 2 ∗ i;
end
```

Geometric Series

$$1 + 1/2 + 1/2^2 + 1/2^3 + 1/2^4 + \ldots = 1/(1 - 1/2) = 2.$$

$$1 + 1/3 + 1/3^2 + 1/3^3 + 1/3^4 + \ldots = 1/(1 - 1/3) = 3/2.$$

$$1 + 2/3 + (2/3)^2 + (2/3)^3 + (2/3)^4 + \ldots = 1/(1 - 2/3) = 3.$$

For r where $0 < r < 1$,

$$1 + r + r^2 + r^3 + r^4 + \ldots = 1/(1 - r).$$

Sample for loops

function `func`(n)

$x \leftarrow 0$;
for $i \leftarrow 2n$ **to** $(3n^2 + 5n)$ **do**
    for $j \leftarrow 1$ **to** $(i^3 + i^2)$ **do**
        $x \leftarrow x + (i - j)$;
    end
end
return (x);