Problem numbers are from the third edition of Sipser’s book. If unsure about which problem to solve, ask. Collaboration is permitted; looking for solutions from external sources (books, the web, material from previous years, etc.) is prohibited. Printed version is preferred, otherwise please make sure your handwriting is readable.

1. Let $C \subseteq \Sigma^*$ be a language. Prove that $C$ is Turing-recognizable iff a decidable language $D$ exists such that $C = \{ x \in \Sigma^* : (\exists y \in \Sigma^*) \langle x, y \rangle \in D \}$.

2. Let $T = \{ \langle M \rangle : M$ is a T.M. that accepts $w$ reversed whenever it accepts $w. \}$. Determine whether $T$ is decidable, undecidable but recognizable or unrecognizable. Prove your answer.

3. Let

$$M = \{ \langle a, b, c, d \rangle : a, b, c \text{ and } p \text{ are binary integers such that } a^b \equiv c \mod p \}.$$ 

Show that $M \in P$. (Note that the most obvious algorithm does not run in polynomial time. Hint: Try first where $b$ is a power of 2.)

4. Prove that the following language is undecidable:

$$A = \{ \langle M \rangle : M \text{ is a TM with running time } O(n) \}.$$ 

5. Prove that the following language is undecidable:

$$A = \{ \langle M \rangle : L(M) \in \text{TIME}(n) \}.$$