Simplicial polytopes that maximize the isotropic constant are highly symmetric

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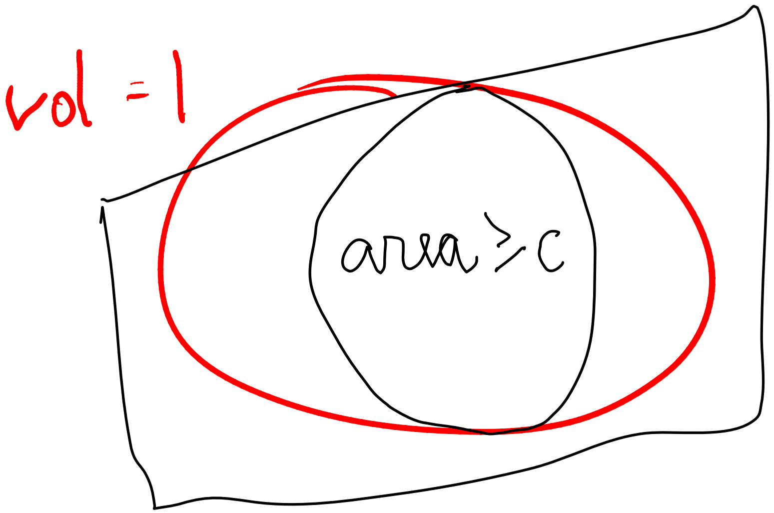
Slicing problem

- Slicing conjecture (Bourgain 1986):
"Every d-dimensional convex body of volume one has a hyperplane section of area at least a universal constant"



More about slicing conjecture

- For K a d-dim convex body, define “isotropic constant”: (A(K) = covariance matrix = $E_{X\in K}(X-\mu)(X-\mu)^T$)

$$L_K^{2d} := \frac{\det A(K)}{(\mathrm{vol}\, K)^2}$$

- Equivalent to slicing conjecture: L_K has a universal constant upper bound.
- Known:
  - $L_K = \Omega(1)$, ellipsoids are the only minimizers.
  - Bourgain: $L_K \le O(\sqrt[4]{d}\log d)$
  - Klartag-Paouris: $L_K \le O(\sqrt[4]{d})$

Connection with Sylvester's problem

- P_K^{d+1}: convex hull of $d+1$ random points in K
- (Sylvester) Maximizer of

$$\frac{E\left(vol\left(P_K^{d+1}\right)\right)}{vol(K)}$$

- Simplex is conjectured maximizer for Sylvester.
- (Slicing) Maximizer of

$$\mathrm{L}_K^{2d} = \frac{\det\left(A(K)\right)}{vol(K)^2} = \frac{d!}{d+1} \frac{E\left(vol\left(P_K^{d+1}\right)^2\right)}{vol(K)^2}$$

Questions-discussion

- Can we understand maximizers of L_K? Is it the simplex? What about 3-D?
- Why slicing instead of Sylvester first? Because second moment (square) should be *much* easier than first (absolute value).

Certain known cases with bounded L_K

- 1-unconditional: $x \in K \Leftrightarrow \mathrm{abs}(x) \in K$ [Bourgain 1986] [V. Milman Pajor 1991]
- Zonoids [Ball]

Other relevant results about L_K

- [Campi Colesanti Gronchi 1999] If K has a non-empty subset of the boundary of class C^2 with positive principal curvatures, then it cannot maximize L_K.
  (more generally, same for pth moment Sylvester's problem)

In other words

- If K has “symmetry X or Y” or if K smooth we have “some understanding”.
- But what if K is not smooth or doesn’t have symmetries? What can we say? Can we argue that maxima must have certain symmetries? After all, the proposed maximizer, (isotropic) simplex, is regular.

(isotropic: $E(X) = 0$, $E(X\,X^T) = I$ for X random in K)

- Inspiration: Similar gap in Mahler’s problem:
  - [Saint Raymond], [Meyer] Cube is minimizer among unconditional bodies, generalized to other symmetries in [Barthe Fradelizi].
  - [Reisner Schütt Werner] improving [Stancu]: If K has a boundary point with positive generalized Gauss curvature, then K cannot minimize the volume product.
    (isotropic: $E(X) = 0$, $E(X\,X^T) = I$ for X random in K)

My result

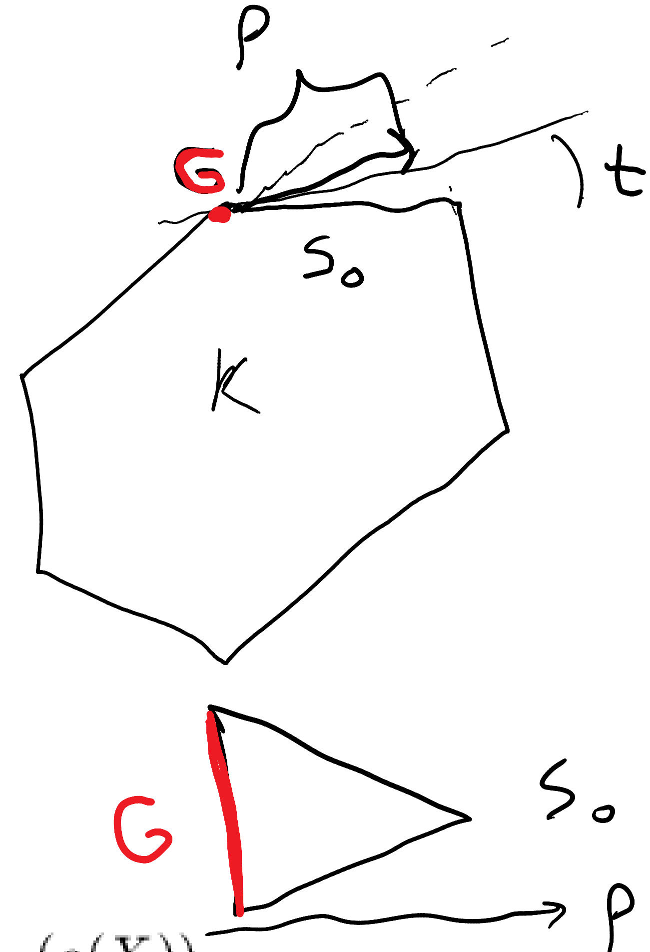
- Let P be a d-dimensional isotropic simplicial polytope that is a maximizer of $P \mapsto L_P$. Then P is *isohedral.*
- Isohedral = For any two facets F, F', there is an orthogonal transformation that maps F to F' and maps P to itself.
- Actually, for two adjacent facets the transformation is just reflection around the hyperplane spanned by their intersection (a $(d-2)$-face)
- Implies that all facets are congruent.

Proof

- Fix a facet $F = conv\{v_1, \dots, v_d\}$, and a facet G of F.
- Consider perturbation: "Hinging" of F around G. "Derivative=0" gives one equation on vertices of F.
- Varying G, gives d equations.

Derivative w.r.t. hinging, isotropic

- t: parameter, angle
- K_t: perturbation of polytope K
- ρ: distance to affine hull of G.
- S_0: facet
- $X \leftarrow D$: random vector in facet S_0 with density proportional to radius $\rho(X)$.

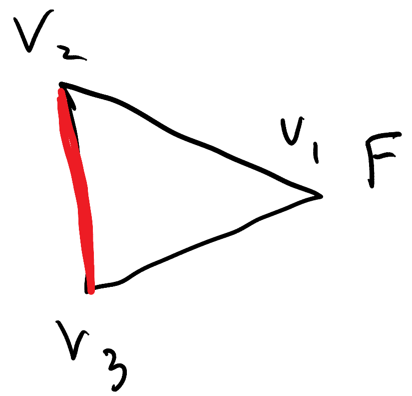


$$\frac{d}{dt} L^{2d}_{K_t} \Big|_{t=0} = \left(\mathbb{E}_{X \leftarrow \mathcal{D}}(\|X\|^2) - d - 2 \right) \frac{\mathbb{E}_{X \in S_0}(\rho(X))}{(\mathrm{vol}\, K)^3}.$$

Resulting equations from explicit derivative

Lemma 3 (First order necessary condition). *Let P be a d-dimensional simplicial polytope that is a critical point of $P \mapsto L_P$. Let F be a facet of P with vertices $v_1, \ldots, v_d$. Then, for $k = 1, \ldots, d$ we have*

$$\sum_{1 \le i \le j \le d} (1 + \delta_{ik} + \delta_{jk}) v_i \cdot v_j = \frac{(d+1)(d+2)^2}{2}.$$



- Easy: for derivative, integrate homogeneous polynomial over arbitrary simplex using formula of [Lasserre Avrachenkov], with $H(x, y) = x \cdot y$:

Theorem 2.1. *Let $x_0, x_1, \ldots, x_n$ be the vertices of an n-dimensional simplex Δ_n. Then, for a symmetric q-linear form $H : (\mathbb{R}^n)^q \to \mathbb{R}$, one has*

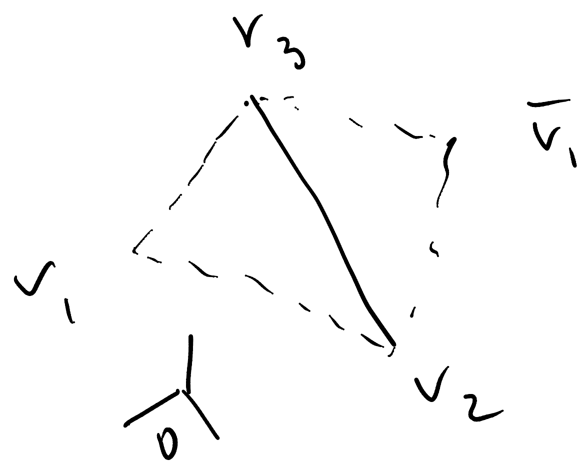
$$\int_{\Delta_n} H(x, x, \ldots, x)\, dx = \frac{\mathrm{vol}(\Delta_n)}{\binom{n+q}{q}} \left[\sum_{0 \le i_1 \le i_2, \ldots, \le i_q \le n} H(x_{i_1}, x_{i_2}, \ldots, x_{i_q}) \right]. \quad (2.3)$$

Now what?

- Idea: d equations on v_1 given $v_2, \ldots, v_d$.
- In 2-D implies regular polygon.

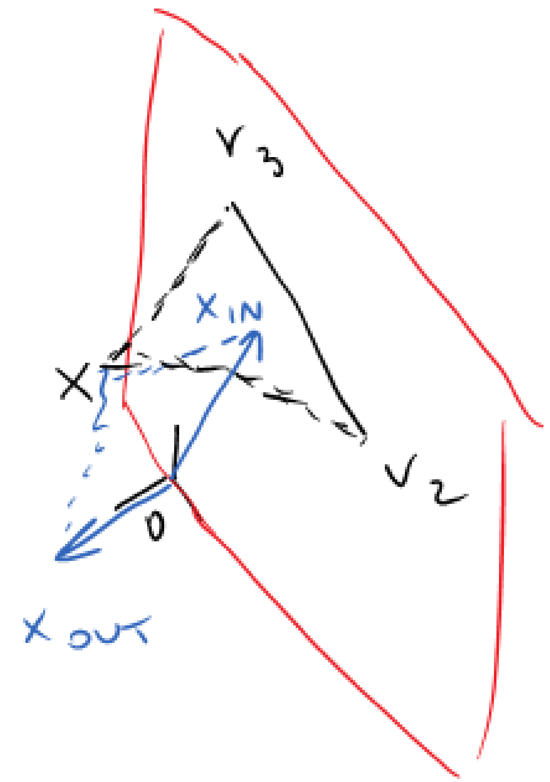
Now what?

- In 3D: "fix edge v_2, v_3", what are the possible adjacent triangular facets?
  Computational experiments suggest only one triangle possible (on each side of edge).
  Would imply adjacent facets are congruent.



Proof of uniqueness

- Rename our unknown v_1 as x, and write $x = x_{IN} + x_{OUT}$,
  $x_{IN} \in span\{v_2, \ldots, v_d\}$,
  $x_{OUT} \in span\{v_2, \ldots, v_d\}^{\perp}$.
- Enough to show:
  - x_{IN} unique.
  - $\|x_{OUT}\|$ unique



Proof of uniqueness

- Rename our unknown v_1 as x, and write $x = x_{IN} + x_{OUT}$, $x_{IN} \in span\{v_2, \dots, v_d\}$, $x_{OUT} \in span\{v_2, \dots, v_d\}^{\perp}$.

$$\frac{(d+1)(d+2)^2}{2} = 3\|x_{IN}\|^2 + 3\|x_{OUT}\|^2 + 2\sum_{i=2}^{d} x_{IN} \cdot v_j + \sum_{2 \leq i \leq j \leq d} v_i \cdot v_j. \quad (1)$$

For $k = 2, \dots, d$ we get

$$\frac{(d+1)(d+2)^2}{2} = \|x_{IN}\|^2 + \|x_{OUT}\|^2 + \sum_{i=2}^{d} x_{IN} \cdot v_j + \sum_{2 \leq i \leq j \leq d} v_i \cdot v_j + x_{IN} \cdot v_k + \sum_{k \leq j \leq d} v_k \cdot v_j + \|v_k\|^2. \quad (2)$$

- $\|x\|^2$ is the only non-linearity. Use (1) to eliminate it from (2):

$$\frac{1}{3} x_{IN} \cdot \sum_{j=2}^{d} v_j + x_{IN} \cdot v_k + \frac{2}{3} \sum_{2 \leq i \leq j \leq d} v_i \cdot v_j + \sum_{k \leq j \leq d} v_k \cdot v_j + \|v_k\|^2 = \frac{2}{3} \frac{(d+1)(d+2)^2}{2}$$

Proof of uniqueness

$$\frac{1}{3}x_{IN}\cdot\sum_{j=2}^{d}v_j+x_{IN}\cdot v_k+\frac{2}{3}\sum_{2\leq i\leq j\leq d}v_i\cdot v_j+\sum_{k\leq j\leq d}v_k\cdot v_j+\|v_k\|^2=\frac{2}{3}\frac{(d+1)(d+2)^2}{2}$$

- In basis $\{v_2, \ldots, v_d\}$ the $(d-1)\times(d-1)$ matrix of the system on x_{IN} is $I+\mathbf{1}\mathbf{1}^T/3$, which is invertible.
- So, x_{IN} unique.
- Given x_{IN}, (1) determines $||x_{OUT}||$ uniquely.