Approximating the Centroid is Hard

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ABSTRACT

Consider the problem of computing the centroid of a convex body in $\mathbb{R}^n$. We prove that if the body is a polytope given as an intersection of half-spaces, then computing the centroid exactly is $\#P$-hard, even for order polytopes, a special case of 0–1 polytopes. We also prove that if the body is given by a membership oracle, then for any deterministic algorithm that makes a polynomial number of queries there exists a body satisfying a roundedness condition such that the output of the algorithm is outside a ball of radius $\sigma/100$ around the centroid, where σ^2 is the minimum eigenvalue of the inertia matrix of the body.

Categories and Subject Descriptors

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General Terms

Algorithms, theory

Keywords

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1. INTRODUCTION

Given a convex body in $\mathbb{R}^n$, the centroid is a basic property that one may want to compute. It is a natural way of representing or summarizing the set with just a single point. There are also diverse algorithms that use centroid computation as a subroutine (for an example, see [2], convex optimization). The following non-trivial property illustrates the power of the centroid: Any hyperplane through the centroid of a convex body cuts it into two parts such that each has a volume that is at least a $1/e$ fraction of the volume of the body.

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There are no known efficient deterministic algorithms for computing the centroid of a convex body exactly. In this paper we will see that this is natural by proving the following result:

THEOREM 1. *It is $\#P$-hard to compute the centroid of an polytope given as an intersection of halfspaces, even if the polytope is an order polytope.*

(Order polytopes are defined in Section 2)

By centroid computation being $\#P$-hard we mean here that for any problem in $\#P$, there is a polynomial time Turing machine with an oracle for centroids of order polytopes that solves that problem.

On the other hand, there are efficient randomized algorithms for approximating the centroid of a convex body given by a membership oracle (See [2]. Essentially, take the average of $O(n)$ random points in the body. Efficient sampling from a convex body is achieved by a random walk, as explained in [6]). We will see that no deterministic algorithm can match this, by proving the following:

THEOREM 2. *There is no polynomial time deterministic algorithm that when given access to a membership oracle of a convex body K such that*

$$\frac{1}{8n} B_n \subseteq K \subseteq n B_n$$

outputs a point at distance $\sigma/100$ of the centroid, where σ^2 is the minimum eigenvalue of the inertia matrix of K.

(The inertia matrix of a convex body is defined in Section 2)

That the centroid is hard to compute is in some sense folklore, but we are not aware of any rigorous analysis of its hardness. The hardness is mentioned in [2] and [4] without proof, for example.

2. PRELIMINARIES

Let $K \subseteq \mathbb{R}^n$ be a compact set with nonempty interior. Let X be a random point in K. The *centroid* of K is the point $c = \mathbb{E}(X)$. The *inertia matrix* of K is the n by n matrix $\mathbb{E}\big((X-c)(X-c)^T\big)$.

For $K \subseteq \mathbb{R}^n$ bounded and a a unit vector, let $w_a(K)$, the width of K along a, be defined as:

$$w_a(K) = \sup_{x \in K} a \cdot x - \inf_{x \in K} a \cdot x.$$

By *canonical directions* in $\mathbb{R}^n$ we mean the set of vectors that form the columns of the n by n identity matrix.

For $a, b, c \in \mathbb{R}$, $a, b > 0$ and $c \geq 1$ we say that *a is within a factor of c of b* iff

$$\frac{1}{c} b \leq a \leq cb.$$

For a partial order $\prec$ of $[n] = \{1, \dots, n\}$, the *order polytope* associated to it is

$$P(\prec) = \{x \in [0,1]^n \ : \ x_i \leq x_j \text{ whenever } i \prec j\}.$$

In [3], it is proven that computing the volume of order polytopes (given the partial order or, equivalently, the facets of the polytope) is $\#P$-complete. We will use this result to prove Theorem 1.

The following well known hardness result for volume approximation combined with a reduction from volume computation to centroid computation will prove Theorem 2.

THEOREM 3 ([1]). *For any deterministic algorithm making at most n^k queries there is a convex body K for which the output is not within a factor*

$$\left(\frac{c_0 n}{k \log n}\right)^{n/2}$$

of $\operatorname{vol} K$*, for some universal constant $c_0 > 0$.*

3. PROOFS

The idea of both proofs is to reduce volume computation to centroid computation, given that it is know in several senses that volume computation is hard.

A basic step in the proofs is the following *key idea*: if a convex body is cut into two pieces, then one can know the ratio between the volumes of the pieces if one knows the centroids of the pieces and of the convex body. Namely, if the body has centroid c and the pieces have centroids c_-, c_+, then the volumes of the pieces are in proportion $\|c - c_-\| / \|c - c_+\|$.

It is known that the volume of a polytope given as an intersection of halfspaces can have a bit-length that is exponential in the length of the input [5]. It is not hard to see that the centroid of a polytope given in that form may also need exponential space. Thus, to achieve a polynomial time reduction from volume to centroid, we need to consider a family of polytopes such that all the centroids that appear in the reduction have a length that is polynomial in the length of the input. To this end we consider the fact that it is $\#P$-hard to compute the volume of order polytopes.

LEMMA 4. *Let P be an order polytope. Then the centroid of P and the volume of P have a bit-length that is polynomial in the bit-length of P.*

PROOF. Call "total order polytope" an order polytope corresponding to a total order. Such a polytope is actually a simplex with 0–1 vertices, its volume is $1/n!$ and its centroid has polynomial bit-length. The set of total order polytopes forms a partition of $[0,1]^n$ into $n!$ parts, and any order polytope is a disjoint union of at most $n!$ total order polytopes. The lemma follows. □

PROOF PROOF (OF THEOREM 1).. Let $P \subseteq [0,1]^n$ be an order polytope, given as a set of halfspaces of the form $H_k = \{x \ : \ x_{i_k} \leq x_{j_k}\}$, $k = 1, \dots, K$. Suppose that we have access to an oracle that can compute the centroid of an order polytope. Then we can compute $\operatorname{vol} P$ in the following way: Consider the sequence of bodies that starts with $[0,1]^n$, and then adds one constraint at a time until we reach P. That is, $P_0 = [0,1]^n$, $P_k = P_{k-1} \cap H_k$. In order to use the *key idea*, for every k, let $Q_k = P_{k-1} \setminus P_k$, compute the centroid c_k of P_k and the centroid d_k of Q_k. We have $P_{k-1} = P_k \uplus Q_k$ and

$$\frac{\operatorname{vol} Q_k}{\operatorname{vol} P_k} = \frac{\|c_{k-1} - d_k\|}{\|c_{k-1} - c_k\|}.$$

Thus,

$$\frac{\operatorname{vol} P_{k-1}}{\operatorname{vol} P_k} = \frac{\|c_{k-1} - d_k\|}{\|c_{k-1} - c_k\|} + 1.$$

This implies, multiplying over all k,

$$\operatorname{vol} P = \prod_{k=1}^{K} \left(\frac{\|c_{k-1} - d_k\|}{\|c_{k-1} - c_k\|} + 1 \right)^{-1}.$$

The reduction costs $2K$ centroid oracle calls. Even though some expressions involve norms, all the intermediate quantities are rational (as the volumes of order polytopes are rational). Moreover, the bit-length of the intermediate quantities is polynomial in n (Lemma 4). □

PROOF PROOF (OF THEOREM 2).. Suppose for a contradiction that there exists an algorithm that finds a point at distance $C\sigma$ of the centroid. Then the following algorithm would approximate the volume in a way that contradicts Theorem 3, for a value of C to be determined. Theorem 3 is actually proved for a family of convex bodies restricted in the following way: We can assume that the body contains the axis-aligned cross-polytope of diameter $2n$ and is contained in the axis-aligned hypercube of side $2n$. Let P be a convex body satisfying that constraint, given as a membership oracle.

Algorithm

1. Let $M = 1$, $i = 0$, $P_0 = P$.

2. For every canonical direction a:

   (a) While $w_a(P) \geq 1$:

      i. $i \leftarrow i + 1$.

      ii. Compute an approximate centroid c_{i-1} of P_{i-1}. Let H be the hyperplane through c orthogonal to a.

      iii. Let P_i be (as an oracle) the intersection of P_{i-1} and the halfspace determined by H containing the origin (if H contains the origin, pick any halfspace).

      iv. Let Q_i be (as an oracle) $P_{i-1} \setminus P_i$.

      v. Compute an approximate centroid d_i of Q_i.

      vi. $M \leftarrow M \frac{\|c_{i-1} - c_i\|}{\|d_i - c_i\|}$

3. Let V be the volume of P_i. Output V/M.

To see that the algorithm terminates, we will show that the "while" loop ends after $O(n \log n)$ iterations. Assuming that $C \leq 1/2$, at every iteration $w_a(P_i)$ decreases at most by a factor of $1/(4n)$ (Lemma 8). Thus, P_i always contains a hypercube of side $1/(4n)$, and $\operatorname{vol} P_i \geq 1/(4n)^n$. Initially, $\operatorname{vol} P_0 \leq (2n)^n$, and every iteration multiplies the volume by a factor of at most $1 - \frac{1}{e} + C$ (Lemma 7). Thus, the algorithm runs for at most

$$\frac{2n \log(n\sqrt{8})}{\log\left(1 - \frac{1}{e} + C\right)^{-1}}$$

iterations.

We will now argue that for all the centroids that the algorithm computes, it knows a ball contained in the corresponding body. Let σ_i^2 be the minimum eigenvalue of the inertia matrix of P_i. Initially, the algorithm knows that P_0 contains a ball of radius $\sqrt{n}$ around the origin. Also, for every i, P_i contains a ball of radius σ_i around the centroid. Because P_i contains a hypercube of side $1/(2n)$, we have that $\sigma_i \geq 1/(2\sqrt{3}n)$. Thus, after we compute c_i, the algorithm knows that P_i contains a ball of radius $(1-C)\sigma_i \geq (1-C)/(2\sqrt{3}n)$ around c_i, and this implies that the algorithm knows that P_{i+1}, Q_{i+1} contain balls of radius $(1-C)/(4\sqrt{3}n)$ around known points.

At step 3, P_i contains the origin and has width at most 1 along all canonical directions. This implies that it is completely contained in the input body, as the input body contains the cross-polytope of diameter $2n$. Thus, the volume of P_i is easy to compute because it is a hypercube that we know explicitly at this point, the intersection of all the halfspaces chosen by the algorithm.

At every cut, $\|c_{i-1} - c_i\|$ is within a constant factor of the true value, as the following argument shows: Let δ_i^2 be the minimum eigenvalue of the inertia matrix of Q_i. Let $\bar{c}_i$, $\bar{d}_i$ be the centroids of P_i, Q_i, respectively. We have that $\|c_i - \bar{c}_i\| \leq C\sigma_i \leq C\sigma_{i-1}$ and $\|d_i - \bar{d}_i\| \leq C\delta_i \leq C\sigma_{i-1}$. That is,

$$\|\bar{c}_{i-1} - \bar{c}_i\| - 2C\sigma_{i-1} \leq \|c_{i-1} - c_i\| \leq \|\bar{c}_{i-1} - \bar{c}_i\| + 2C\sigma_{i-1}$$

and we also have that the true distance satisfies (by Lemma 5)

$$\|\bar{c}_{i-1} - \bar{c}_i\| \geq \sigma_{i-1}/2.$$

Thus, the estimate satisfies:

$$(1-4C)\|\bar{c}_{i-1} - \bar{c}_i\| \leq \|c_{i-1} - c_i\| \leq (1+4C)\|\bar{c}_{i-1} - \bar{c}_i\|.$$

A similar argument shows:

$$(1-4C)\|\bar{d}_i - \bar{c}_i\| \leq \|d_i - c_i\| \leq (1+4C)\|\bar{d}_i - \bar{c}_i\|.$$

Thus, M, as an estimate of $V/\operatorname{vol} P$, is within a factor of

$$\left(\frac{1+4C}{1-4C}\right)^{\frac{2n\log(n\sqrt{8})}{\log(1-\frac{1}{e}+C)^{-1}}}$$

of the true value, and so is the estimate of the volume, V/M, with respect to $\operatorname{vol} P$. The choice of $C = 1/100$ would give the contradiction. $\square$

Lemma 5 (centroid versus σ). *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Let a be a unit vector. Let $K_+ = K \cap \{x : a \cdot x \geq 0\}$. Let X be random in K. Let c be the centroid of K_+. Let $\sigma^2 = \mathbb{E}\big((X \cdot a)^2\big)$. Then*

$$c \cdot a \geq \sigma/2.$$

Proof. Let X_+ be random in K_+, let $K_- = K \setminus K_+$, let X_- be random in K_-. Let $\sigma_+^2 = \mathbb{E}((X_+ \cdot a)^2)$, $\sigma_-^2 = \mathbb{E}((X_- \cdot a)^2)$. Lemma 6 implies $c \cdot a \geq \sigma_+/\sqrt{2}$. To relate σ and σ_+, we observe that σ is between σ_+ and σ_-, and we use Lemma 6 again:

$$\sigma_+ \geq \mathbb{E}(X_+ \cdot a) = -\mathbb{E}(X_- \cdot a) \geq \sigma_-/\sqrt{2}.$$

This implies $\sigma_+ \geq \sigma/\sqrt{2}$ and the lemma follows. $\square$

The following is a particular case of Lemma 5.3 (c) in [7].

Lemma 6 ($\mathbb{E}(X)$ versus $\mathbb{E}(X^2)$). *Let X be a non-negative random variable with logconcave density function $f : \mathbb{R}^+ \to \mathbb{R}$. Then*

$$2(\mathbb{E}\, X)^2 \geq \mathbb{E}(X^2).$$

The next lemma follows from the proof of Theorem 1 in [2]:

Lemma 7 (volume lemma). *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin, let σ^2 be the minimum eigenvalue of the inertia matrix of K, let $c \in \mathbb{R}^n$. Let a be a unit vector. Let $K_+ = K \cap \{x : a \cdot x \geq a \cdot c\}$. Then*

$$\operatorname{vol} K_+ \geq \left(\frac{1}{e} - \frac{|c \cdot a|}{\sigma}\right) \operatorname{vol} K.$$

Lemma 8 (width lemma). *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin, let σ^2 be the minimum eigenvalue of the inertia matrix of K, let $c \in \mathbb{R}^n$. Let a be a unit vector. Let $K_+ = K \cap \{x : a \cdot x \geq a \cdot c\}$. Then*

$$w_a(K_+) \geq \left(1 - \frac{|c \cdot a|}{\sigma}\right) \frac{w_a(K)}{2n}.$$

Proof. Consider an ellipsoid E centered at the origin such that $E \subseteq K \subseteq nE$ (Löwner-John pair). We have that $\frac{1}{2} w_a(E) \geq \sigma$. Then

$$\begin{aligned} w_a(K^+) &\geq \frac{1}{2} w_a(E) - |c \cdot a| \\ &\geq \left(1 - \frac{|c \cdot a|}{\sigma}\right) \frac{1}{2} w_a(E) \\ &\geq \left(1 - \frac{|c \cdot a|}{\sigma}\right) \frac{1}{2n} w_a(K). \end{aligned}$$

$\square$

4. CONCLUSION

We proved two hardness results for the computation of the centroid of a convex body. Some open problems suggested by this work are the following:

- Find a substantial improvement of Theorem 2, that is, is the centroid hard to approximate even within a ball of radius superlinear in σ?
- Prove a lower bound on the query complexity of any randomized algorithm that approximates the centroid. A possible approach may be given by the lower bound for volume approximation in [8].

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5. REFERENCES

[1] I. Bárány and Z. Füredi. Computing the volume is difficult. *Discrete & Computational Geometry*, 2:319–326, 1987.

[2] D. Bertsimas and S. Vempala. Solving convex programs by random walks. *J. ACM*, 51(4):540–556, 2004.

[3] G. Brightwell and P. Winkler. Counting linear extensions. *Order*, 8(3):225–242, 1991.

[4] M. J. Kaiser, T. L. Morin, and T. B. Trafalis. Centers and invariant points of convex bodies. In *Applied geometry and discrete mathematics*, volume 4 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 367–385. Amer. Math. Soc., Providence, RI, 1991.

[5] J. Lawrence. Polytope volume computation. *Mathematics of Computation*, 57(195):259–271, 1991.

[6] L. Lovasz and S. Vempala. Fast Algorithms for Logconcave Functions: Sampling, Rounding, Integration and Optimization. *Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)-Volume 00*, pages 57–68, 2006.

[7] L. Lovász and S. Vempala. The geometry of logconcave functions and sampling algorithms. *Random Structures and Algorithms*, 2006. To appear.

[8] L. Rademacher and S. Vempala. Dispersion of mass and the complexity of randomized geometric algorithms. In *FOCS '06: Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)*, pages 729–738, Washington, DC, USA, 2006. IEEE Computer Society.