Approximating the Centroid is Hard

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Centroid computation

- Given $K \subseteq \mathbb{R}^n$ compact, find its centroid, given by $\mathbb{E}(X)$, where X is random in K.

Our results

- Two hardness results, similar to hardness for volume computation.

	volume	centroid
exact	$\#P$-hard (Dyer, Frieze) (Brightwell, Graham)	$\#P$-hard (this paper)
approximate, oracle	Elekes; Bárány and Füredi polytime $\implies$ error of $n^{n/2}$	this paper

- We only deal with deterministic algorithms.

Our results: exact

- It is $\#P$-hard to compute the centroid of a convex body given as an intersection of halfspaces (even when the input is restricted to 0–1 polytopes, even order polytopes).

Our results: approximate and oracle

- There is no polynomial time algorithm that, when given access to a well-rounded convex body K by a membership oracle, finds a point within distance $\sigma/100$ of the centroid, where σ^2 is the minimum eigenvalue of the inertia matrix of K.
  (Inertia matrix: For $X \in K$ random, $\mathbb{E}(XX^T)$, i.e., covariance matrix of X.)

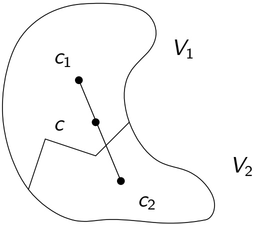
  Small correction to paper, roundness condition is

$$\frac{1}{17n^2} B_n \subseteq K \subseteq 2n^2 B_n.$$

Proof idea

- Reductions from volume problem to centroid, key idea: knowing c, c_1, c_2 we know

$$\frac{V_1}{V_2} = \frac{\|c_2 - c\|}{\|c_1 - c\|}$$



Proof idea: exact

- For hardness of exact centroid:

Theorem (Brightwell, Graham (1991))

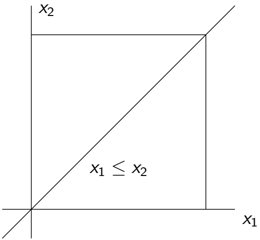
It is $\#P$-hard to compute the volume of order polytopes.

- For a partial order $\prec$ of $[n] = \{1, \ldots, n\}$, the *order polytope* associated to it is

$$P(\prec) = \{x \in [0,1]^n \ : \ x_i \leq x_j \text{ whenever } i \prec j\}.$$

Why order polytopes: because then centroid is "strongly" $\#P$-hard, i.e., even when the numbers in the input polytope are small.

Proof idea: exact



- Add constraints "$\{x_i \leq x_j\}$" one by one to get sequence of polytopes $P_1, P_2, \ldots, P_k$, using "key idea" to keep track of ratios of volumes: $\operatorname{vol} P_{i+1} / \operatorname{vol} P_i$. $P_1 = [0,1]^n$, $\operatorname{vol} P_1 = 1$, want $\operatorname{vol} P_k$, given by

$$\operatorname{vol} P_k = \operatorname{vol} P_1 \frac{\operatorname{vol} P_2}{\operatorname{vol} P_1} \cdots \frac{\operatorname{vol} P_k}{\operatorname{vol} P_{k-1}}.$$

- Another reason for order polytopes: intermediate centroids in the reduction better have polynomial bit-length; not true for arbitrary polytopes, but true for order polytopes.

Proof idea: approximate

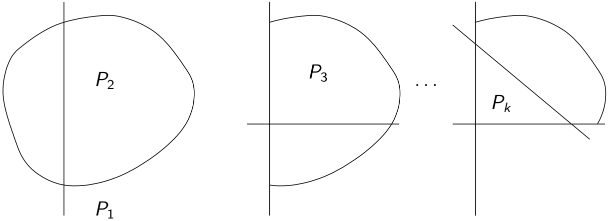
- For hardness of approximate centroid:

Theorem (Elekes; Bárány and Füredi)

Any algorithm that makes a polynomial number of membership queries fails to approximate the volume up to a factor of $\sim n^{n/2}$.

Proof idea: approximate

- Make a sequence of cuts to reach a shape whose volume is easy to compute, while keeping track of the ratios of the volumes, given by centroids and "key idea".



- Reduction approximates volume as a product of the form

$$\mathrm{vol}\, P_1 = \mathrm{vol}\, P_k \frac{\mathrm{vol}\, P_{k-1}}{\mathrm{vol}\, P_k} \cdots \frac{\mathrm{vol}\, P_1}{\mathrm{vol}\, P_2}.$$

  where $\mathrm{vol}\, P_k$ is easy, P_1 is input.

- Difficulty: need to keep volume's ratios bounded and need to keep each piece well-rounded.

Proof idea: approximate

Why dependence on σ?

- Error (distance) of approximate centroids should depend on σ to approximate ratios $\operatorname{vol} P_{i-1} / \operatorname{vol} P_i$ up to a multiplicative constant.
- Ratios need to be bounded by $\operatorname{poly}(n)$, this happens if cut near true centroid, "near" depends on σ.

Discussion

- Proved: two hardness results for exact and approximate computation of the centroid.
- Open: Is the centroid hard to approximate in a ball of radius superlinear in σ?
- Open: Lower bound for randomized approximation of the centroid, maybe along the lines of lower bound for the volume by R. and Vempala.