Dispersion of Mass and the Complexity of Randomized Algorithms II

ICMS

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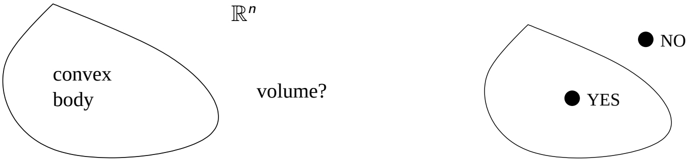
Volume computation



- $\#P$-hard for polytopes given as a list of vertices or halfspaces.
- There is a family of polytopes having rational halfspaces such that the volume is a rational a/b with b of length exponential in the input length.
- Standard model: membership oracle. Randomization provably helps.
- Because of hardness results, we forget exact computation and consider approximations up to a multiplicative factor.

Known bounds for volume

- Deterministic lower bound:
  Elekes; Báráni, Füredi: polynomial number of queries $\Longrightarrow$ error of a factor of $\sim n^{n/2}$.
- Upper bound: random walks:

Dyer, Frieze, Kannan 1991	n^{23}
Lovász, Siminovits 1990	n^{16}
Applegate, Kannan 1990	n^{10}
Lovász 1990	n^{10}
Dyer, Frieze 1991	n^{8}
Lovász, Simonovits 1993	n^{7}
Kannan et al. 1997	n^{5}
Lovász, Vempala 2003	n^{4}

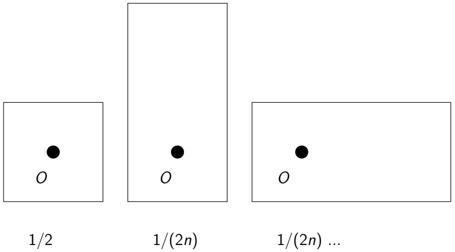
  This and the deterministic lower bound imply that randomization provably helps.
- The complexity of a randomized approximation of the volume depends on roundedness of the input. Our lower bound holds even for well-rounded convex bodies.

Yao's lemma

- Yao's lemma:
  The probability of failure of a randomized algorithm on the worst input is at least the probability of failure of the best deterministic algorithm against some distribution.

Easy lower bound for randomized algorithms

- Lower bound: $\Omega(n)$.
  - Distribution: Axis-aligned cube or brick.
  - Result: less than $n-1$ queries $\implies$ fails to approximate volume up to a constant factor with probability at least $1/2n$



First idea for improved lower bound

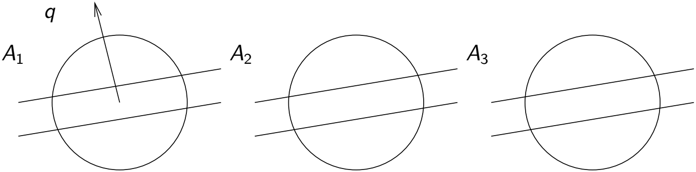
- Instead of axis-aligned, consider all rotations of cube and brick, $\sim n^2$ "degrees of freedom".
- But given that our proof is in the flavor of information theory, we prefer the space of transformations to be a subset of Euclidean space with the usual measure. Instead of rotations, we allow all linear transformations of a cube, suitably restricted to satisfy roundness condition.
  $\implies$ more variability in volume.

Our results: lower bounds, volume, determinant

- Theorem (volume): Any randomized algorithm needs $\Omega(n^2/\log n)$ queries to approximate the volume up to a constant factor with probability $1 - 1/\operatorname{poly}(n)$ (even well-rounded).
- Actually, it is hard for parallelepipeds of the form: for $A \in \mathbb{R}^{n \times n}$
$$\{x \in \mathbb{R}^n \ : \ (\forall i)|A_i \cdot x| \leq 1\}$$
Oracle: given $x \in \mathbb{R}^n$, decide whether $(\forall i)|A_i \cdot x| \leq 1$.
Theorem (determinant): Any randomized algorithm needs $\Omega(n^2/\log n)$ queries to approximate $|\det A|$ up to a constant factor with probability $1 - 1/\operatorname{poly}(n)$.

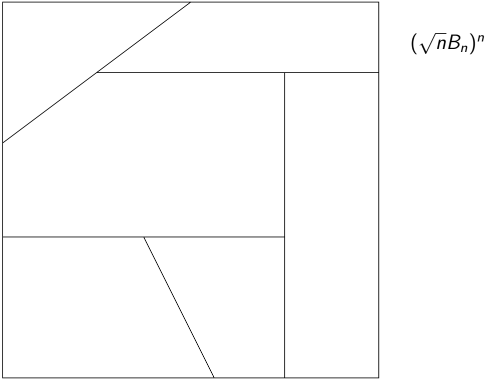
Proof of the volume lower bound

- Yao's lemma: enough to prove a lower bound for deterministic algorithms against a distribution.
- Distribution: Parallelepipeds "$\|Ax\|_\infty \leq 1$" given by matrices A with each row uniformly and independently from $\sqrt{n}B_n$.
- A deterministic algorithm can be seen as a decision tree $\implies$ partition of the input space $((\sqrt{n}B_n)^n)$.



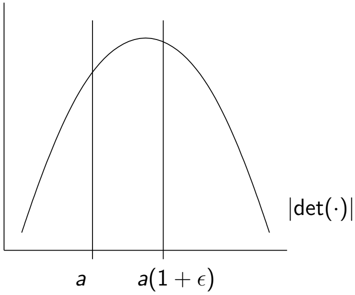
Proof of the volume lower bound

- After making all its queries, what the algorithm knows is exactly on which part of the partition of $(\sqrt{n}B_n)^n$ we are



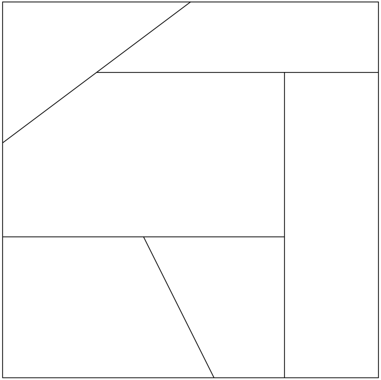
Proof of the volume lower bound

- Failure of the algorithm $\Leftrightarrow$ dispersion of $|\det(\cdot)|$ on most parts.
- Constant dispersion: there is at least $1/\operatorname{poly}(n)$ mass outside of any interval of constant multiplicative length (i.e. of the form $[a, a(1+\epsilon)]$).



Proof of the volume lower bound

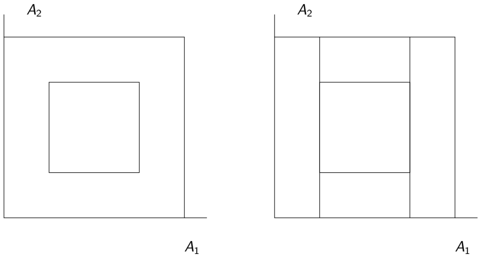
- Leaf $\leftrightarrow$ part. $\#$ parts $= \#$ leafs $\leq 2^{\text{depth}}$.



$(\sqrt{n}B_n)^n$

Proof of the volume lower bound

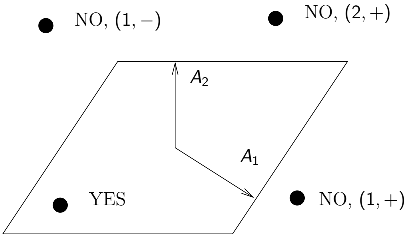
- Parts can be assumed to be "product sets along rows".



Implemented by modified oracle.

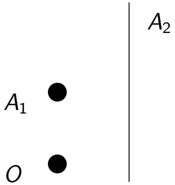
Proof of the volume lower bound

- Modified oracle: given q, if $\|Aq\|_\infty \leq 1$ output YES, otherwise output NO and least index among violated constraints and also side of the violation. $(2n+1)$-ary tree.



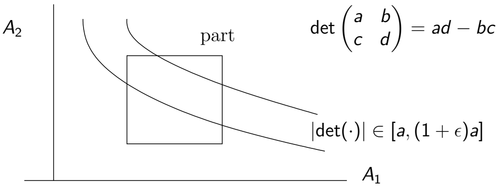
Proof of the volume lower bound

- Warm up: What if a part is a "product set along rows" and the determinant on that part has a fixed value?
  Then, regions for each row are
  - a point,
  - a line parallel to the span of the previous point,
  - a plane parallel to the span of the previous regions ...



Proof of the volume lower bound

- Intuition: shape of parts is very different from level sets of $|\det(\cdot)|$, as parts can be assumed to be "product sets along rows". If $|\det(\cdot)|$ is not dispersed in a part, then the part is small $\implies$ many parts $\implies$ tree of large depth $(n^2/\log n)$.



Main lemma for volume lower bound

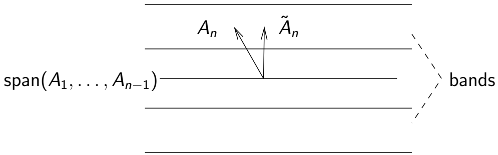
- (dispersion of the determinant)
  For any partition of B_n^n into $\leq 2^{n^2}$ parts that are "product sets along rows", for half of the parts in measure we have

$$\Pr\Big(|\det X| \notin [u, u(1+c)]\Big) \geq \frac{1}{2^7 n^6}$$

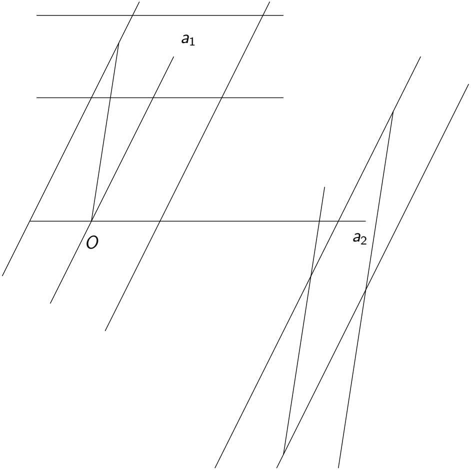
  for any u, for X a random point (matrix) in the part, and for c a universal constant.

Proof of lemma

- Proof idea: Assume (for a contradiction) that a part $P = \prod_i P_i$ is such that, for most matrices in it, $|\det(\cdot)|$ fits in a small interval. Write $|\det(A)| = \prod_i \|\tilde{A}_i\|$, where $\tilde{A}_i$ is the projection of A_i to the space orthogonal to $A_1, \dots, A_{i-1}$. Then
$$\|\tilde{A}_n\| \in \left[a, a(1+\epsilon)\right] \prod_{i=1}^{n-1} \|\tilde{A}_i\|^{-1}.$$
Thus, for most choices of $A_1, \dots, A_{n-1}$, most of $A_n \in P_n$ is forced to be contained in a “band”. A refinement of this argument gives a band on P_i for each pair i, j.



Proof of lemma: pairs of bands



Proof of lemma

- $\mathrm{vol}((\sqrt{n}B_n)^n) \geq 2^{n^2} + 2^{n^2}$ parts.
  $\implies$ half of the parts (in measure) have volume at least $1/2$ each.
- $|\det(\cdot)|$ concentrated + "product set of matrices"
  $\implies$ bands
  $\implies$ volume of corresponding part less that $1/2$.
  Contradiction.

Conclusion

We proved:

- an $n^2/\log n$ lower bound for the query complexity of approximating the volume of a convex body up to a constant factor by a randomized algorithm.

Conclusion

Open problems:

- Complexity of approximating the volume: matching upper and lower bounds. Same with a separation oracle.