On the monotonicity of the expected volume of a random simplex

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Sylvester's problem

- 4 random points: what is the probability that they are in convex position?
- P_n^K: convex hull of n random points in convex body K.
- Which convex bodies K are extremal for

$$\frac{\mathbb{E}\,\mathrm{vol}\,P_n^K}{\mathrm{vol}\,K}$$

Questions by Meckes and Reitzner

$$K \subseteq M \implies \mathbb{E}\,\mathrm{vol}\,P_n^K \leq c^d \mathbb{E}\,\mathrm{vol}\,P_n^M$$

- M’s weak conjecture: for n=d+1 (random simplex) there exists some c.
- M’s strong conjecture: for n=d+1 and c=1
- R’s question: for arbitrary n and c=1.

Connection with slicing

- Slicing conjecture: every d-dimensional convex body of volume one has a hyperplane section of area >=c for some universal c.
- Equivalent: for K a d-dimensional convex body

$L_K := (\det A(K)/(\operatorname{vol} K)^2)^{1/2d}$ has a universal upper bound.

- M's weak conjecture ⇔ slicing

M's weak conjecture:
$\exists c > 0$ such that for any pair K, M of d-dimensional convex bodies

$$K \subseteq M \implies \mathbb{E} \operatorname{vol} P_{d+1}^K \leq c^d \mathbb{E} \operatorname{vol} P_{d+1}^M$$

Main result

- About M's strong conjecture:
  - True in dimension 1,2
  - False in dimension >=4
  - Strong numerical evidence for falsity in dim. 3.

M's strong conjecture:
For any pair K, M of d-dimensional convex bodies

$$K \subseteq M \overset{?}{\Longrightarrow} \mathbb{E}\,\mathrm{vol}\,P_{d+1}^{K} \leq \mathbb{E}\,\mathrm{vol}\,P_{d+1}^{M}$$

Busemann-Petty

$$(\forall\theta \in S_{d-1})\,\mathrm{vol}_{d-1}(K \cap \theta^{\perp}) \leq \mathrm{vol}_{d-1}(L \cap \theta^{\perp})$$

$$\overset{?}{\Longrightarrow}\ \mathrm{vol}\,K \leq c\,\mathrm{vol}\,L$$

- Classical: when c=1, true iff dim<5
- $\exists$ c $\Leftrightarrow$ slicing
- Similarity with our problem:
  - Dimension-dependent answer, connection with slicing
- Difference with our problem:
  - Ours has “elementary” solution, no Fourier analysis.

Question by Vempala

$$K \subseteq M \overset{?}{\implies} \det A(K) \leq c^d \det A(M)$$

- A(K) = covariance matrix of K
- Original question: for c=1.

Question by Vempala

$$K \subseteq M \overset{?}{\Longrightarrow} \det A(K) \leq c^d \det A(M)$$

- $\exists$ c $\Leftrightarrow$ slicing:

$$\det A(K) / \operatorname{vol}(K)^2 = L_K^{2d} \quad *$$

$\Leftarrow$: easy from * and bounded isotropic constant

$\Rightarrow$:

1. Let K,M be two convex bodies. Assumption with * imply $L_K \leq c^{1/2}$ d(K,M) L_M :
   By affine invariance, w.l.o.g. K,M are in position such that K $\subseteq$ M $\subseteq$ d(K,M)K. Implies vol M $\leq$ d(K,M)^d vol K.
2. Use Klartag's isomorphic slicing problem:
   - given K,ε, there exists M s.t. d(K,M)≤ 1+ε and

$$L_M \leq 1/\sqrt{\epsilon}$$

Question by Vempala

- Connection with random simplexes: (μ(K)=centroid of K)

$$\det A(K) = d!\mathbb{E}_{X_i \in K}\Big((\operatorname{vol}\operatorname{conv}\mu(K), X_1, \ldots, X_d)^2\Big)$$
$$= \frac{d!}{d+1}\mathbb{E}\Big((\operatorname{vol} P^K_{d+1})^2\Big),$$

- I.e. "second moment" version of Meckes's question.

Second result

- We also solve Vempala's question for c=1
  - True in dimension 1,2
  - False in dimension >=3
- Our solution to Vempala's question inspired our solution to Meckes's strong conjecture.

Vempala's question (for $c = 1$):

$$K \subseteq L \overset{?}{\Longrightarrow} \det A(K) \leq \det A(L)$$

Solution to Vempala’s question

- Intuition: extreme point near the centroid

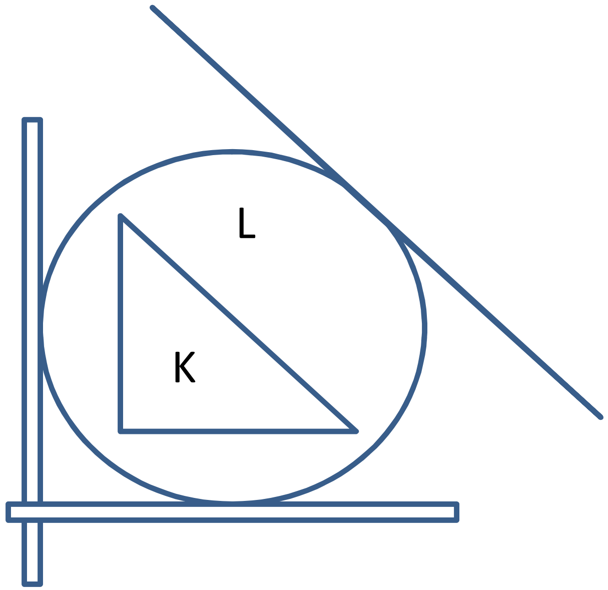
- E (random simplex using L\K) < E(... using L)

Solution to Vempala's question

- Idea: Monotonicity of K $\mapsto$ det A(K) holds for dimension *d* iff for every K $\subseteq$ L there is a non-increasing "path" of convex bodies from L to K. We will:
  - define "path",
  - compute derivative along path, and
  - study sign of derivative.

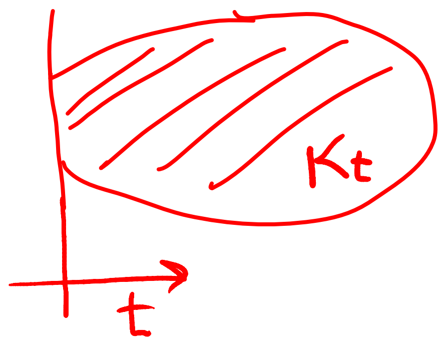
Solution to Vempala's question

- W.l.o.g. K is a polytope (by continuity, if there is a counterexample, then there is one where K is a polytope)
- "Path": "push" hyperplanes parallel to facets of K "in", one by one.



Solution to Vempala's question

- det A(·) continuous along path, piecewise continuously differentiable.
- Enough to compute derivative with respect intersection with moving halfspace.
- Enough to compute derivative in isotropic position (sign of derivative is invariant under affine transformations).



$$K_t = K \cap \{x : a^T x \le t\}$$

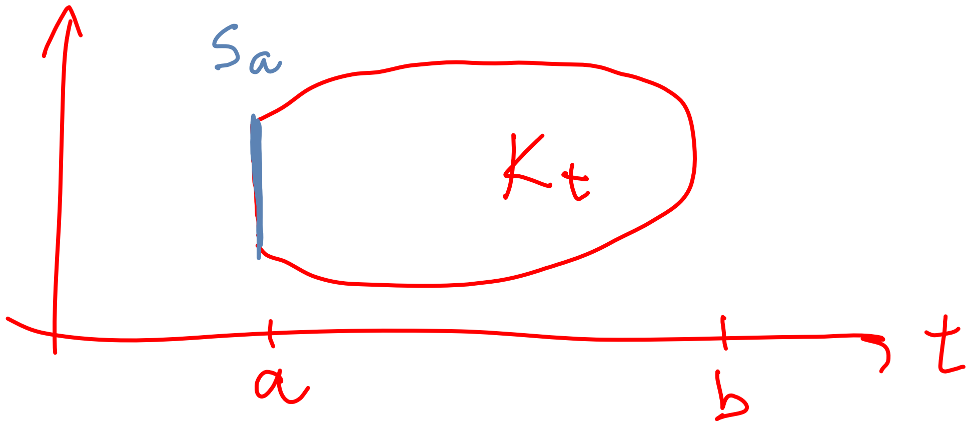
$$\frac{d}{dt} \det A(K_t)$$

Solution to Vempala's question

- Simple value of derivative in isotropic position:

Proposition. *Let* $K \subseteq \mathbb{R}^d$ *be an* *isotropic convex body.* *Let* $v \in \mathbb{R}^d$ *be a unit vector. Let* $a = \inf_{x \in K} v \cdot x$, $b = \sup_{x \in K} v \cdot x$. *Let* $H_t = \{x \in \mathbb{R}^d : v \cdot x \geq t\}$. *Let* $K_t = K \cap H_t$, $S_t = K \cap \mathrm{bdry}\, H_t$. *Then*

$$\frac{d}{dt} \det A(K_t)\Big|_{t=a} = \left(d - \mathbb{E}_{X \in S_a} \|X\|^2\right) \frac{\mathrm{vol}_{d-1} S_a}{\mathrm{vol}\, K}.$$



Solution to Vempala's question

- Derivative implies dimension dependent condition:

Lemma. *Monotonicity under inclusion of* $K \mapsto \det A(K)$ *holds for some dimension* d *iff for any isotropic convex body* $K \subseteq \mathbb{R}^d$ *we have*

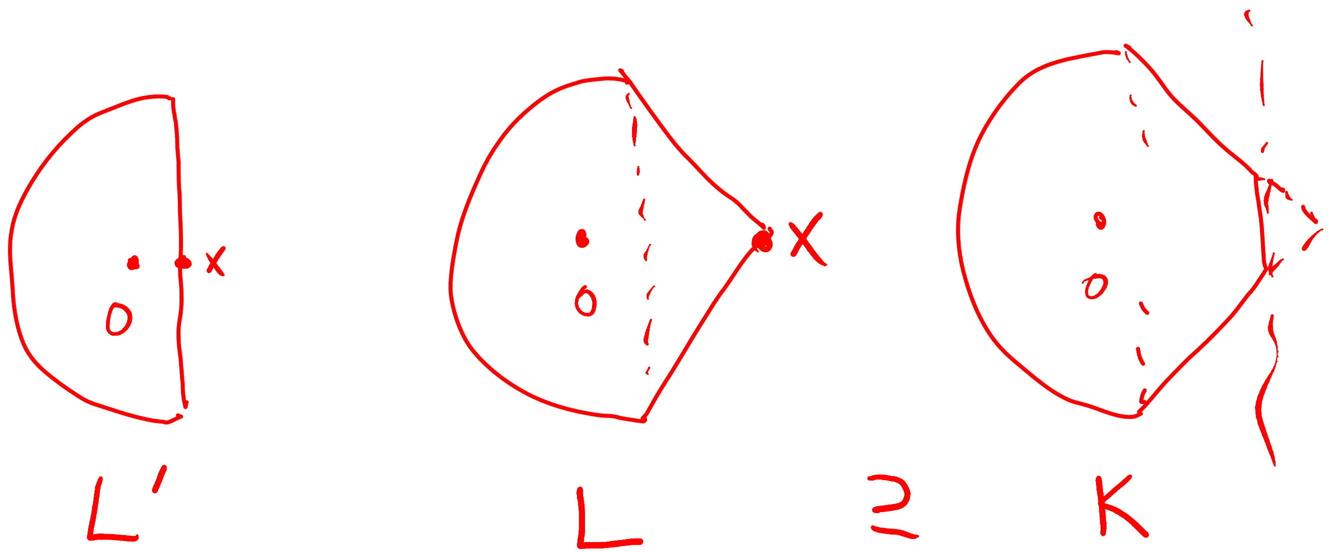
$$\sqrt{d} B_d \subseteq K. \quad \bigstar$$

- Proof:
  - "if part": condition $\bigstar$ implies negative derivative along path.

$$\frac{d}{dt} \det A(K_t)\Big|_{t=a} = \left(d - \mathbb{E}_{X \in S_a} \|X\|^2\right) \frac{\mathrm{vol}_{d-1} S_a}{\mathrm{vol}\, K}.$$

Proof (cont'd)

- "Only if" part:
  - There is an isotropic convex body L' with a boundary point x at distance <$d^{1/2}$ from the origin.
  - By an approximation argument can assume x is extreme point (keeping isotropy and distance condition) of new body L.
  - Positive derivative as one pushes hyperplane "in" at x a little bit.
  - If K is "L truncated near x", det A(K) > det A(L).



When does the condition hold?

- Condition: for any isotropic convex body K

$$\sqrt{d}B_d \subseteq K.$$

- Milman-Pajor, Kannan-Lovász-Simonovits: For any isotropic convex body K:

$$\sqrt{\frac{d+2}{d}}B_d \subseteq K \subseteq \sqrt{d(d+2)}B_d$$

  and this is best possible.
- I.e., condition fails iff d $\geq$ 3.

Solution to Meckes's strong conjecture

- Argument parallels case of det A(K):
  - same path between K and L (push hyperplanes in), same derivative (i.e. with respect to moving hyperplane)
- Our derivative is a special case Crofton's differential equation:

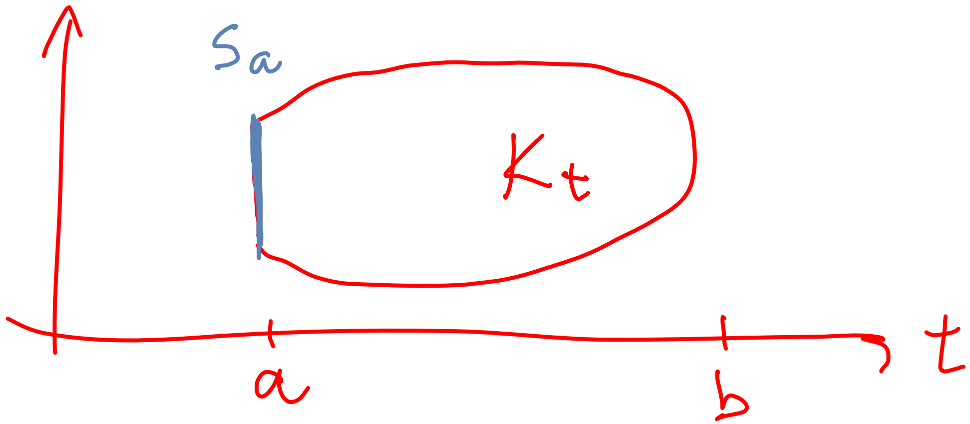
CROFTON, M. W. Article "Probability". *Encyclopaedia Britannica*, 9th edn. (1885).

(from Kendall and Moran, "Geometrical Probability", 1963)

Derivative

Proposition (general derivative, Crofton). *Let $K \subseteq \mathbb{R}^d$ be a convex body. Let $v \in \mathbb{R}^d$ be a unit vector. Let $a = \inf_{x \in K} v \cdot x$, $b = \sup_{x \in K} v \cdot x$. Let $H_t = \{x \in \mathbb{R}^d : v \cdot x \geq t\}$. Let $K_t = K \cap H_t$, $S_t = K \cap \operatorname{bdry} H_t$. Let $f : (\mathbb{R}^d)^k \to \mathbb{R}$ be a symmetric continuous function. Let $X_1, \ldots, X_k$ be random points in K. Then*

$$\frac{d}{dt} \mathbb{E} f(X_1, \ldots, X_k) \Big|_{t=a} = k\Big(\mathbb{E} f(X_1, \ldots, X_k) - \mathbb{E}\big(f(X_1, \ldots, X_k) \mid X_1 \in S_a\big)\Big) \frac{\operatorname{vol}_{d-1} S_a}{\operatorname{vol} K}$$



Solution to Meckes's strong conjecture

- Dimension dependent condition (same proof):

Lemma. *Monotonicity under inclusion of*

$$K \mapsto \mathbb{E}_{X_i \in K} \operatorname{vol} \operatorname{conv} X_0, \ldots, X_d$$

holds for some dimension d *iff for any convex body* $K \subseteq \mathbb{R}^d$ *and any* $x \in \operatorname{bdry} K$ *and* $X_0, \ldots, X_d$ *random in* K *we have*

$$\mathbb{E} \operatorname{vol} \operatorname{conv} X_0, X_1, \ldots, X_d \leq \mathbb{E} \operatorname{vol} \operatorname{conv} x, X_1, \ldots, X_d.$$

When does the condition hold?

Fails for $d \geq 4$: $K =$ half ball, $x = 0$, $L =$ ball of same volume as K.

$$\begin{aligned} \mathbb{E}\,\mathrm{vol}\,P^K_{d+1} &\geq \mathbb{E}\,\mathrm{vol}\,P^L_{d+1} \qquad \text{(Blaschke-Groemer)} \\ &= \frac{1}{d!}\left(\frac{\kappa_{d+1}}{\kappa_d}\right)^{d+1} \frac{\kappa_{d(d+2)}}{\kappa_{(d+1)^2}} \frac{1}{\omega_{d+1}} \qquad \text{(known)} \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbb{E}_K \,\mathrm{vol}\,\mathrm{conv}(0, X_1, \ldots, X_d) &= \mathbb{E}_{B_d} \,\mathrm{vol}\,\mathrm{conv}(0, X_1, \ldots, X_d) \qquad \text{(symmety)} \\ &= \frac{1}{d!}\left(\frac{\kappa_{d+1}}{\kappa_d}\right)^{d} \frac{2}{\omega_{d+1}} \qquad \text{(known)} \end{aligned}$$

Combine to get

$$\frac{\mathbb{E}_K \,\mathrm{vol}\,\mathrm{conv}(0, X_1, \ldots, X_d)}{\mathbb{E}\,\mathrm{vol}\,P^K_{d+1}} \leq 2\frac{\sqrt{d+2}}{d+1}$$

When does the condition hold?

- true for d=1, easy (directly, without the condition)

When does the condition hold?

True for $d = 2$:

Theorem (Blaschke). *Among all 2-dimensional convex bodies,*

$$\frac{\mathbb{E}_{X_i \in K}(\operatorname{vol} \operatorname{conv}(X_0, X_1, X_2))}{\operatorname{vol}(K)} \leq \frac{1}{12}$$

with equality iff K is a triangle.

Lemma. *Let $K \subseteq \mathbb{R}^2$ be a convex body and let $x \in \operatorname{bdry} K$. Then*

$$\frac{\mathbb{E}_{X_1, X_2 \in K}(\operatorname{vol} \operatorname{conv} x, X_1, X_2)}{\operatorname{vol} K} \geq \frac{8}{9\pi^2}. \tag{1}$$

Proof idea: Worst case, K is half ellipse, x its "center". Then Steiner symmetrization, Blaschke's Schüttelung (shaking), symmetrization around x, Busemann's inequality.

When does the condition hold?

- d=3? The proof doesn’t handle it, but numerical integration strongly suggests “false” (same counterexample).

Open questions

- Higher moments?
- (Reitzner) What about more than d+1 points?
- (technical) 3-D case
- Weak conjecture via Crofton’s differential equation?