On the monotonicity of the expected volume of a random simplex

Luis Rademacher
Computer Science and Engineering
The Ohio State University
Sylvester’s problem

• 4 random points: what is the probability that they are in convex position?

• $P_n^K$: convex hull of $n$ random points in convex body $K$.

• Which convex bodies $K$ are extremal for

$$\frac{\mathbb{E} \text{ vol } P_n^K}{\text{vol } K}$$
Questions by Meckes and Reitzner

\[ K \subseteq M \implies \mathbb{E} \, \text{vol} \, P_n^K \leq c^d \mathbb{E} \, \text{vol} \, P_n^M \]

• M’s weak conjecture: for \( n=d+1 \) (random simplex) there exists some \( c \).
• M’s strong conjecture: for \( n=d+1 \) and \( c=1 \)
• R’s question: for arbitrary \( n \) and \( c=1 \).
Connection with slicing

- Slicing conjecture: every $d$-dimensional convex body of volume one has a hyperplane section of area $\geq c$ for some universal $c$.
- Equivalent: for $K$ a $d$-dimensional convex body
  \[ L_K := \left( \frac{\det A(K)}{(\text{vol } K)^2} \right)^{1/2d} \] has a universal upper bound.
- M’s weak conjecture $\iff$ slicing
  M’s weak conjecture:
  \[ \exists c > 0 \text{ such that for any pair } K, M \text{ of } d\text{-dimensional convex bodies} \]
  \[ K \subseteq M \implies \mathbb{E} \text{ vol } P_{d+1}^K \leq c^d \mathbb{E} \text{ vol } P_{d+1}^M \]
Main result

• About M’s strong conjecture:
  – True in dimension 1,2
  – False in dimension >=4
  – Strong numerical evidence for falsity in dim. 3.

M’s strong conjecture:
For any pair $K, M$ of $d$-dimensional convex bodies

\[ K \subseteq M \implies \mathbb{E} \text{vol } P_{d+1}^K \leq \mathbb{E} \text{vol } P_{d+1}^M \]
Busemann-Petty

\[(\forall \theta \in S_{d-1}) \ vol_{d-1}(K \cap \theta^\perp) \leq \ vol_{d-1}(L \cap \theta^\perp)\]

\[\implies vol\ K \leq c \ vol\ L\]

- Classical: when \(c=1\), true iff \(\text{dim}<5\)
- \(\exists c \iff \text{slicing}\)
- Similarity with our problem:
  - Dimension-dependent answer, connection with slicing
- Difference with our problem:
  - Ours has “elementary” solution, no Fourier analysis.
Question by Vempala

\[ K \subseteq M \quad \Rightarrow \quad \det A(K) \leq c^d \det A(M) \]

• A(K) = covariance matrix of K
• Original question: for c=1.
Question by Vempala

\[ K \subseteq M \implies \det A(K) \leq c^d \det A(M) \]

• \( \exists c \iff \text{slicing:} \)

\[
\frac{\det A(K)}{\text{vol}(K)^2} = L_K^{2d} \quad * \\
\iff \text{easy from } * \text{ and bounded isotropic constant}
\]

\[ \implies: \]

1. Let \( K, M \) be two convex bodies. Assumption with * imply \( L_K \leq c^{1/2} \text{d}(K,M) L_M \):

   By affine invariance, w.l.o.g. \( K, M \) are in position such that \( K \subseteq M \subseteq \text{d}(K,M)K \). Implies \( \text{vol } M \leq \text{d}(K,M)^d \text{ vol } K \).

2. Use Klartag’s isomorphic slicing problem:

   • given \( K, \varepsilon \), there exists \( M \) s.t. \( \text{d}(K,M) \leq 1 + \varepsilon \) and

   \[ L_M \leq 1/\sqrt{\varepsilon} \]
Question by Vempala

• Connection with random simplexes: \((\mu(K) = \text{centroid of } K)\)

\[
\det A(K) = d! \mathbb{E}_{X_i \in K} \left( (\text{vol conv } \mu(K), X_1, \ldots, X_d)^2 \right)
\]

\[
= \frac{d!}{d+1} \mathbb{E} \left( (\text{vol } P^K_{d+1})^2 \right),
\]

• I.e. “second moment” version of Meckes’s question.
Second result

• We also solve Vempala’s question for $c=1$
  – True in dimension 1,2
  – False in dimension $\geq 3$

• Our solution to Vempala’s question inspired our solution to Meckes’s strong conjecture.

Vempala’s question (for $c = 1$):

$$K \subseteq L \implies \det A(K) \leq \det A(L)$$
Solution to Vempala’s question

• Intuition: extreme point near the centroid

• $E$ (random simplex using $L \setminus K) < E(\ldots$ using $L)$
Solution to Vempala’s question

• Idea: Monotonicity of $K \mapsto \det A(K)$ holds for dimension $d$ iff for every $K \subseteq L$ there is a non-increasing “path” of convex bodies from $L$ to $K$. We will:
  – define “path”,
  – compute derivative along path, and
  – study sign of derivative.
Solution to Vempala’s question

• W.l.o.g. K is a polytope (by continuity, if there is a counterexample, then there is one where K is a polytope)
• “Path”: “push” hyperplanes parallel to facets of K “in”, one by one.
Solution to Vempala’s question

• $\det A(\cdot)$ continuous along path, piecewise continuously differentiable.
• Enough to compute derivative with respect intersection with moving halfspace.
• Enough to compute derivative in isotropic position (sign of derivative is invariant under affine transformations).

$$K_t = K \cap \{x : a^T x \leq t\}$$

$$\frac{d}{dt} \det A(K_t)$$
Solution to Vempala’s question

- Simple value of derivative in isotropic position:

**Proposition.** Let $K \subseteq \mathbb{R}^d$ be an **isotropic convex body**. Let $v \in \mathbb{R}^d$ be a unit vector. Let $a = \inf_{x \in K} v \cdot x$, $b = \sup_{x \in K} v \cdot x$. Let $H_t = \{x \in \mathbb{R}^d : v \cdot x \geq t\}$. Let $K_t = K \cap H_t$, $S_t = K \cap \text{bdry} H_t$. Then

$$\frac{d}{dt} \det A(K_t) \bigg|_{t=a} = \left(d - \mathbb{E}_{X \in S_a} \|X\|^2\right) \frac{\text{vol}_{d-1} S_a}{\text{vol} K}.$$
Solution to Vempala’s question

• Derivative implies dimension dependent condition:

**Lemma.** Monotonicity under inclusion of $K \mapsto \det A(K)$ holds for some dimension $d$ iff for any isotropic convex body $K \subseteq \mathbb{R}^d$ we have

$$\sqrt{d}B_d \subseteq K.$$  

• Proof:

  – “if part”: condition ⭐️ implies negative derivative along path.

$$\frac{d}{dt} \det A(K_t)\bigg|_{t=a} = \left(d - \mathbb{E}_{X \in S_a} \|X\|^2\right) \frac{\text{vol}_{d-1} S_a}{\text{vol } K}.$$
“Only if” part:
- There is an isotropic convex body $L'$ with a boundary point $x$ at distance $d^{1/2}$ from the origin.
- By an approximation argument can assume $x$ is extreme point (keeping isotropy and distance condition) of new body $L$.
- Positive derivative as one pushes hyperplane “in” at $x$ a little bit.
- If $K$ is “$L$ truncated near $x$”, $\det A(K) > \det A(L)$.
When does the condition hold?

• Condition: for any isotropic convex body $K$

$$\sqrt{d}B_d \subseteq K.$$  

• Milman-Pajor, Kannan-Lovász-Simonovits: For any isotropic convex body $K$:

$$\frac{\sqrt{d+2}}{d} B_d \subseteq K \subseteq \sqrt{d(d+2)}B_d$$

and this is best possible.

• I.e., condition fails iff $d \geq 3.$
Solution to Meckes’s strong conjecture

• Argument parallels case of det A(K):
  – same path between K and L (push hyperplanes in),
    same derivative (i.e. with respect to moving
    hyperplane)

• Our derivative is a special case Crofton’s
differential equation:

(from Kendall and Moran, “Geometrical
Probability”, 1963)
**Proposition** (general derivative, Crofton). Let $K \subseteq \mathbb{R}^d$ be a convex body. Let $v \in \mathbb{R}^d$ be a unit vector. Let $a = \inf_{x \in K} v \cdot x$, $b = \sup_{x \in K} v \cdot x$. Let $H_t = \{ x \in \mathbb{R}^d : v \cdot x \geq t \}$. Let $K_t = K \cap H_t$, $S_t = K \cap \text{bdry } H_t$. Let $f : (\mathbb{R}^d)^k \to \mathbb{R}$ be a symmetric continuous function. Let $X_1, \ldots, X_k$ be random points in $K$. Then

$$\frac{d}{dt} \left. \mathbb{E} f(X_1, \ldots, X_k) \right|_{t=a} = k \left( \mathbb{E} f(X_1, \ldots, X_k) - \mathbb{E} (f(X_1, \ldots, X_k) \mid X_1 \in S_a) \right) \frac{\text{vol}_{d-1} S_a}{\text{vol } K}$$
Solution to Meckes’s strong conjecture

• Dimension dependent condition (same proof):

Lemma. Monotonicity under inclusion of

\[ K \mapsto \mathbb{E}_{X_i \in K} \text{vol conv } X_0, \ldots, X_d \]

holds for some dimension \( d \) iff for any convex body \( K \subseteq \mathbb{R}^d \) and any \( x \in \text{bdry } K \) and \( X_0, \ldots, X_d \) random in \( K \) we have

\[ \mathbb{E} \text{ vol conv } X_0, X_1, \ldots, X_d \leq \mathbb{E} \text{ vol conv } x, X_1, \ldots, X_d. \]
When does the condition hold?

Fails for \( d \geq 4 \): \( K = \) half ball, \( x = 0 \), \( L = \) ball of same volume as \( K \).

\[
\mathbb{E} \text{vol } P_d^{K} \geq \mathbb{E} \text{vol } P_d^{L} \quad \text{(Blaschke-Groemer)}
\]

\[
= \frac{1}{d!} \left( \frac{\kappa_{d+1}}{\kappa_d} \right)^{d+1} \frac{\kappa_{d(d+2)}}{\kappa_{(d+1)^2}} \frac{1}{\omega_{d+1}} \quad \text{(known)}
\]

On the other hand,

\[
\mathbb{E}_K \text{vol conv}(0, X_1, \ldots, X_d) = \mathbb{E}_{B_d} \text{vol conv}(0, X_1, \ldots, X_d) \quad \text{(symmetry)}
\]

\[
= \frac{1}{d!} \left( \frac{\kappa_{d+1}}{\kappa_d} \right)^d \frac{2}{\omega_{d+1}} \quad \text{(known)}
\]

Combine to get

\[
\frac{\mathbb{E}_K \text{vol conv}(0, X_1, \ldots, X_d)}{\mathbb{E} \text{vol } P_d^{K}} \leq 2 \frac{\sqrt{d+2}}{d+1}
\]
When does the condition hold?

- true for $d=1$, easy (directly, without the condition)
When does the condition hold?

True for $d = 2$:

**Theorem** (Blaschke). *Among all 2-dimensional convex bodies,*

\[
\mathbb{E}_{X_i \in K} \left( \frac{\text{vol conv}(X_0, X_1, X_2)}{\text{vol}(K)} \right) \leq \frac{1}{12}
\]

*with equality iff $K$ is a triangle.*

**Lemma.** *Let $K \subseteq \mathbb{R}^2$ be a convex body and let $x \in \text{bdry } K$. Then*

\[
\mathbb{E}_{X_1, X_2 \in K} \left( \frac{\text{vol conv } x, X_1, X_2}{\text{vol } K} \right) \geq \frac{8}{9\pi^2}. \tag{1}
\]

**Proof idea:** Worst case, $K$ is half ellipse, $x$ its “center”. Then Steiner symmetrization, Blaschke’s Schüttelung (shaking), symmetrization around $x$, Busemann’s inequality.
When does the condition hold?

• d=3? The proof doesn’t handle it, but numerical integration strongly suggests “false” (same counterexample).
Open questions

• Higher moments?
• (Reitzner) What about more than $d+1$ points?
• (technical) 3-D case
• Weak conjecture via Crofton’s differential equation?