# Minimal Partitioning into Product Sets

Luis Rademacher and Xuancheng Shao

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**Abstract**

While proving a lower bound of $n^2/\log n$ membership queries to approximate the volume of a convex body in $\mathbb{R}^n$, [4] proved the following: Given $q_1, \ldots, q_m \in \mathbb{R}^n$, let $Q_i = \{a \in \mathbb{R}^n : |a \cdot q_i| \leq 1\}^n$ for $1 \leq i \leq m$. Then the set $\mathbb{R}^{n^2} - \bigcup_{i=1}^m Q_i$ can be partitioned into no more than $n^m$ product sets of the form $\prod_{j=1}^n A_j$, $A_j \in \mathbb{R}^n$. The following question was implicitly left open: for $m = O(n^2)$, can this be improved to $c^{n^2}$? If so, the lower bound for the volume would improve to $n^2$. Here we give a negative answer to the open question, that is, a lower bound of $n^{\Omega(n^2)}$ for the partitioning problem. More generally, we give tight bounds for all values of the parameters of the partitioning problem, by proving covering lower bounds.

## 1 Introduction

Consider the problem of computing the volume of a convex body in $\mathbb{R}^n$ given by a membership oracle. On the one hand, there are randomized algorithms that approximate the volume of a well-rounded convex body up to any given constant factor in time[1] $O^*(n^4)$ and with probability $1 - 1/\operatorname{poly}(n)$, [1, 2]. On the other hand, [4] proves a lower bound of $\Omega(n^2/\log n)$ to approximate within some constant factor with probability $1 - 1/n$. The proof involves giving an upper bound to the size of parts of the partition of the input space induced by the decision tree associated to an algorithm. This implies that any such partition has many parts, and therefore gives a lower bound on the height of the tree (which corresponds to the query complexity of the algorithm). While proving this, they prove the following (simple) fact: Given $q_1, \ldots, q_m \in \mathbb{R}^n$, let $Q_i = \{a \in \mathbb{R}^n : |a \cdot q_i| \leq 1\}^n$ for $1 \leq i \leq m$. Then the set $\mathbb{R}^{n^2} - \bigcup_{i=1}^m Q_i$ can be partitioned into no more than $n^m$ product sets of the form $\prod_{j=1}^n A_j$, $A_j \in \mathbb{R}^n$ (The original problem was to partition $A^n - \bigcup_{i=1}^m Q_i$ where $A$ is the ball of radius $\sqrt{n}$; in the context of this paper, nothing changes substantially as long as $A \subseteq \mathbb{R}^n$ and 0 is in the interior of $A$, for simplicity we set $A = \mathbb{R}^n$). The following question was implicitly left open: for $m = O(n^2)$, can this be improved

[1] The $*$ in $O^*$ means that logarithmic factors are ignored.

to $c^{n^2}$ for some $c > 0$? If so, the lower bound for the volume would improve to $\Omega(n^2)$.

Our main result (Theorem 12) gives a negative answer to the open question, that is, a lower bound of $n^{\Omega(n^2)}$ for that partitioning problem. More generally, we give tight bounds for all ranges of the parameters of the partitioning problem, by proving covering lower bounds. For some range of the parameters (in particular, for $m = n^2$), the upper bound on the number of parts holds even when the sets $Q_i$ are products of arbitrary sets (i.e. $Q_i = \prod_{j=1}^{n} S_i^{(j)}$, $S_i$ arbitrary) instead of powers of bands. For completeness, and to introduce some of the aspects of the proofs in a simpler setting, we analyze this (easy) case of arbitrary $Q_i$'s in detail, showing that one can always partition into $n^m$ product sets, and this is attained for some choice of the $S_i^{(j)}$'s (Proposition 4).

The rest of the paper is organized as follows: In Section 2 we give an optimal bound for the case where the $Q_i$'s are products of arbitrary sets (the "general" case). In Section 3, we give tight bounds for the case where the $Q_i$'s are powers of bands (what we call the "ball" case); this section uses some of the tools described in Section 2.

# 2 The general partitioning and covering problems

In this section, we will deal with the following problem:

**Problem:** Let $m, n$ be positive integers. Define $f_p(n,m)$ (resp. $f_c(n,m)$) to be the smallest value of $k$, such that for any sets $X^{(1)}, \ldots, X^{(n)}$ and subsets $S_i^{(j)} \subseteq X^{(j)}$ $(1 \leq i \leq m)$, there always exists a partition (resp. covering) by $k$ product sets (that is, of the form: $\prod_{j=1}^{n} Y^{(j)}$ for some $Y^{(j)} \subseteq X^{(j)}$) of the set

$$P = \prod_{j=1}^{n} X^{(j)} - \bigcup_{i=1}^{m} \prod_{j=1}^{n} S_i^{(j)}. \tag{1}$$

From now on, we will call this problem the general partitioning or covering problem, for convenience. In Section 2.1 we will prove that $f_p(n,m) \leq n^m$. In Section 2.2 we will prove that $f_c(n,m) \geq n^m$. Since $f_p(n,m) \geq f_c(n,m)$, we can conclude that $f_p(n,m) = f_c(n,m) = n^m$.

## 2.1 Upper bound for the general partitioning problem

In this Section, we will present an explicit way to partition the set $P$ by induction on $n$.

**Proposition 1.** $f_p(n,m) \leq n^m$.

*Proof.* We will prove this by induction on $n$. If $n = 1$, the fact that $f_p(1,m) \leq 1$ is trivial. Suppose that $f_p(n-1,m) \leq (n-1)^m$ for all $m$. Note that

$$P = \bigcup_{I \subseteq [m]} \left[ \left( \bigcap_{i \in I} S_i^{(1)} - \bigcup_{i \notin I} S_i^{(1)} \right) \times \left( \prod_{j=2}^{n} X^{(j)} - \bigcup_{i \in I} \prod_{j=2}^{n} S_i^{(j)} \right) \right], \tag{2}$$

where the union is taken over all $2^m$ possible subsets $I$ of $[m]$. From Equation (2) we can see that the set $P$ can be partitioned into $2^m$ subsets (some of them may be empty), each of them having the form $Y \times \left( \prod_{j=2}^{n} X^{(j)} - \bigcup_{i \in I} \prod_{j=2}^{n} S_i^{(j)} \right)$ for some $I \subseteq [m]$, where $Y$ is a subset of $X^{(1)}$. By induction hypothesis, this subset can be partitioned into $(n-1)^{|I|}$ product sets. Therefore, we have

$$\begin{aligned} f_p(n,m) &\leq \sum_{I \subseteq [m]} (n-1)^{|I|} \\ &= \sum_{k=0}^{m} \binom{m}{k} (n-1)^k \\ &= n^m. \end{aligned} \tag{3}$$

□

## 2.2 Lower bound for the general covering problem

In this section, we will show that at least $n^m$ product sets are needed in the general covering problem in the worst case. In order to show this, we wish to select a maximum number of points in $P$ such that any two of them cannot be covered by the same product set. This means that the number of product sets needed in any covering is at least the number of points chosen in this way. In order to select these points, we define a concept of *good* partitions of the set $[m]$ such that each selected point is *incident* to a good partition. Note that here the order of the parts of a partition of $[m]$ matters.

**Definition 2.** *A partition $I = (I_1, \ldots, I_n)$ of the set $[m]$ ($[m] = \{1, \ldots, m\}$) is* good *if there exists a point $x = (x_1, x_2, \ldots, x_n)$ such that $x_j \in \bigcap_{i \notin I_j} S_i^{(j)} - \bigcup_{i \in I_j} S_i^{(j)}$ for any $1 \leq j \leq n$. We say that the point $x$ is incident to the partition $I$.*

The following lemma shows that points belonging to different good partitions of $[m]$ cannot be covered by the same product set. This implies that $f_c(n,m)$ is at least the number of good partitions of $[m]$.

**Lemma 3.** *Let $I = (I_1, I_2, \ldots, I_n)$ and $J = (J_1, J_2, \ldots, J_n)$ be two different good partitions of the set $[m]$. For any two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ that are incident to $I$ and $J$, respectively, they cannot be covered by the same product set in any partition of $P$.*

*Proof.* Suppose for a contradiction that $x$ and $y$ are covered by the same product set $U$. Since $I$ and $J$ are different partitions of $[m]$, there exists some $i \in [m]$, such that $i \in I_k$ and $i \in J_\ell$ with $k \neq \ell$. Without loss of generality, we can assume that $k < \ell$. Since $x, y \in U$, we have

$$z = (x_1, \ldots, x_{k-1}, y_\ell, x_{k+1}, \ldots, x_n) \in U.$$

However, since $i \in I_k$, $x_j \in S_i^{(j)}$ for $j \neq k$. Since $i \in J_\ell$, $y_\ell \in S_i^{(k)}$. This shows that $z \in \prod_{j=1}^n S_i^{(j)}$, a contradiction.

□

**Proposition 4.** $f_c(n, m) \geq n^m$.

*Proof.* We choose sets $X^{(1)}, \ldots, X^{(n)}$ and subsets $S_i^{(j)} \subseteq X^{(j)} (1 \leq i \leq m)$, such that for any $1 \leq j \leq n$ and $I \subseteq [m]$, we have $\bigcap_{i \notin I} S_i^{(j)} - \bigcup_{i \in I} S_i^{(j)} \neq \emptyset$. Therefore, any partition of the set $[m]$ is good. Because of Lemma 3, $f_c(n, m)$ is at least the number of good partitions of $[m]$, which is $n^m$. Thus we conclude that $f_c(n, m) \geq n^m$.

□

Combining the results of Proposition 1, Proposition 4 and the fact that $f_p(n, m) \geq f_c(n, m)$, we get $f_p(n, m) = f_c(n, m) = n^m$. The upper bound here will be used in Section 3.

# 3 The ball partitioning and covering problems

In this section, we study the following problem:

**Problem:** Let $m, n, d$ be positive integers. Define $h_p(m, n, d)$ (resp. $h_c(m, n, d)$) to be the smallest number $p$ that satisfies the following condition: for any $q_1, q_2, \ldots, q_m \in \mathbb{R}^d$, and $S_i = \{x \in \mathbb{R}^d : |x \cdot q_i| \leq 1\}$ $(1 \leq i \leq m)$, there exist $p$ sets $U_1, U_2, \ldots, U_p$, each of the form $U_i = \prod_{j=1}^n V_i^{(j)}$ where $V_i^{(j)} \subseteq \mathbb{R}^d$, such that they form a partition (resp. covering) of the set $P$ defined by $P = \prod_{j=1}^n \mathbb{R}^d - \bigcup_{i=1}^m S_i^n$.

For convenience, we shall call this problem the ball partitioning (covering) problem (Although this problem has nothing to do with balls, we still use this term since it came from a problem about balls mentioned in the introduction). We will study the behavior of the function $h_p(m, n, d)$ and $h_c(m, n, d)$. In Section 3.1, we will derive two upper bounds for $h_p(m, n, d)$. One of them is suitable for small values of $m$, the other is more appropriate when $m$ becomes large. In Section 3.2 and Section 3.3, we will derive lower bounds for $h_c(m, n, d)$. Once again, the two cases that $m$ is not very large and that $m$ becomes much larger than $n$ and $d$ will be treated differently. We obtain tight bounds up to some constants in the exponent.

## 3.1 Two upper bounds

If we consider the ball partitioning problem as a special case of the general partitioning problem, we immediately get the following corollary of Proposition 1:

**Corollary 5.**

$$h_p(m, n, d) \leq n^m. \tag{4}$$

Note that in Equation (4), the upper bound is not related to the parameter $d$. Now consider the upper bound in another way. Suppose that the boundaries of $S_1, S_2, \ldots, S_m$ partition the space $\mathbb{R}^d$ into $k$ regions $V_1, V_2, \ldots, V_k$ (the boundary points are not relevant for the current discussion: either they are assigned in some way to one of the adjacent regions, or partitions from now on are "coverings of closed parts having pairwise disjoint interiors"). Then the family of sets $\mathcal{F} = \{V_{i_1} \times V_{i_2} \times \cdots \times V_{i_n} : 1 \leq i_1, i_2, \ldots, i_n \leq k\}$ is a partition of $\mathbb{R}^{nd}$. Since each $S_i^n$ is the union of some sets in $\mathcal{F}$, the region defined by $P$ can be partitioned into some of the sets in $\mathcal{F}$. Thus, we have $h_p(m, n, d) \leq k^n$. Now we will consider an upper bound for $k$. Note that the boundary of each $S_i$ is the union of two hyperplanes (defined by $x \cdot q_i = 1$ and $x \cdot q_i = -1$ respectively) in $\mathbb{R}^d$ if $q_i \neq 0$. Therefore, $k$ is at most $g(2m, d)$, where $g(m, d)$ is the maximum number of regions that $\mathbb{R}^d$ can be divided into by $m$ hyperplanes. We have (see [3]):

**Theorem 6.** *For any $m$ and $d$, let $g(m, d)$ denote the largest number of regions that $\mathbb{R}^d$ can be divided into by $m$ hyperplanes. Then,*

$$g(m, d) = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{d}. \tag{5}$$

**Corollary 7.**

$$h_p(m, n, d) \leq g(2m, d)^n = \left[\binom{2m}{0} + \binom{2m}{1} + \cdots + \binom{2m}{d}\right]^n. \tag{6}$$

Therefore, we have two different upper bounds for the function $h_p(m, n, d)$ described in Equations (4) and (6). It is easy to see that Equation (6) is a better upper bound when $m$ is large, and Equation (4) is better when $m$ is small compared to $n$ and $d$.

## 3.2 Cyclic polytopes

Now we turn to the lower bound problem of $h_c(m, n, d)$. We wish to use the results of Lemma 3. For that purpose, we need to find some good partitions $I = (I_1, I_2, \ldots, I_n)$ of the set $[m]$, in which $\bigcap_{i \notin I_j} S_i - \bigcup_{i \in I_j} S_i$ is nonempty for every $j$. We introduce the following definition for convenience:

**Definition 8.** *A subset $I$ of $[m]$ is said to be a cell if the set $\bigcap_{i \notin I} S_i - \bigcup_{i \in I} S_i$ is nonempty.*

Therefore, all the subsets in a good partition are cells. We first consider a sufficient condition for $I \subseteq [m]$ to be a cell. In our case, $S_i$ is defined by the condition $|x \cdot q_i| \leq 1$. For simplicity, we shall first consider $S_i$ to be defined by $x \cdot q_i \leq 1$. Thus, $S_i$ is just a half space defined by a hyperplane in dimension $d$. If we take the $q_i$'s as the $m$ vertices of a polytope of dimension $d$, then $I$ is a cell if the points $q_i$, $i \in I$, are the vertices of a facet of this polytope. To see this, let $a \cdot x = 1$ be the equation that defines this facet where $a \in \mathbb{R}^d$, and $a \cdot q_j < 1$ for all $j \notin I$. Then the point $ca \in \bigcap_{i \notin I} S_i - \bigcup_{i \in I} S_i$, where $c$ is less than but very close to 1. In order for us to have maximal choices for cells, we need to find a polytope with $m$ vertices that has a maximum number of facets. This polytope turns out to be a cyclic polytope [5].

Cyclic polytopes can be constructed by taking the convex hull of $m > d$ points on the moment curve in $\mathbb{R}^d$ (the curve defined by $x(t) = (t, t^2, \ldots, t^d)$ for $t \in \mathbb{R}$). Thus, we can choose $q_i = (t_i, t_i^2, \ldots, t_i^d)$ for some $t_1 < t_2 < \cdots < t_m$. The exact values of $t_i$ will be selected later. Gale's evenness criterion is used to determine whether $d$ points on the moment curve determine a facet of the cyclic polytope:

**Theorem 9.** *If $1 \leq i_1 < i_2 < \cdots < i_d \leq m$, then $q_{i_1}, q_{i_2}, \ldots, q_{i_d}$ determine a facet if and only if the number of indices in $\{i_1, i_2, \ldots, i_d\}$ lying between any two indices not in that set is even.*

A proof of this theorem is given in [5]. As a result, any subset $I$ of $[m]$ which has order $d$, and satisfies Gale's evenness criterion must be a cell. However, recall that we have assumed that $S_i$ is defined by $x \cdot q_i \leq 1$ instead of $|x \cdot q_i| \leq 1$ as it should be. Therefore, Gale's evenness criterion does not necessarily imply that $I$ is a cell in the real case. However, this problem can be solved by choosing $t_i$ suitably instead of arbitrarily: $t_i = m + i$. The reason of this choice will be explicit in the proof of Lemma 10. In this way, we can guarantee that Gale's evenness criterion still implies that $I$ is a cell even for the real definition of $S_i$. Furthermore, since we wish to select $n$ cells which form a partition of the set $[m]$, the restriction implied by Gale's evenness criterion that each cell must have order $d$ may be too strong. Taking this into account, we shall prove the following lemma, which uses the same idea of the proof of Gale's evenness criterion, and is more suitable for our purpose.

**Lemma 10.** *Suppose $\ell \leq d$ and $\ell$ is even. Define $I = \bigcup_{k=1}^{\ell/2} [i_{2k-1}, i_{2k}]$ for some $1 \leq i_1 < i_2 < \cdots < i_\ell \leq m$. Then $I$ is a cell.*

*Proof.* For any $x_1, x_2, \ldots, x_\ell \in \mathbb{R}$, consider the determinant

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & t_{i_1} & \cdots & t_{i_\ell} \\ x_2 & t_{i_1}^2 & \cdots & t_{i_\ell}^2 \\ \vdots & \vdots & & \vdots \\ x_\ell & t_{i_1}^\ell & \cdots & t_{i_\ell}^\ell \end{pmatrix} = a_1 x_1 + a_2 x_2 + \cdots + a_\ell x_\ell + a_0,$$

where

$$a_0 = \det \begin{pmatrix} t_{i_1} & \cdots & t_{i_\ell} \\ t_{i_1}^2 & \cdots & t_{i_\ell}^2 \\ \vdots & & \vdots \\ t_{i_1}^\ell & \cdots & t_{i_\ell}^\ell \end{pmatrix} = \left( \prod_{k=1}^{\ell} t_{i_k} \right) \left[ \prod_{1 \le s < t \le \ell} (t_{i_t} - t_{i_s}) \right].$$

Let $v = (a_1, a_2, \ldots, a_\ell, 0, \ldots, 0) \in \mathbb{R}^d$. We prove that for any $j \notin I$,

$$\left| a_1 t_j + a_2 t_j^2 + \cdots + a_\ell t_j^\ell \right| < a_0 \tag{7}$$

In fact, we have

$$\begin{aligned} a_1 t_j + a_2 t_j^2 + \cdots + a_\ell t_j^\ell &= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_j & t_{i_1} & \cdots & t_{i_\ell} \\ t_j^2 & t_{i_1}^2 & \cdots & t_{i_\ell}^2 \\ \vdots & \vdots & & \vdots \\ t_j^\ell & t_{i_1}^\ell & \cdots & t_{i_\ell}^\ell \end{pmatrix} - a_0 \\ &= \left[ \prod_{k=1}^{\ell} (t_{i_k} - t_j) \right] \left[ \prod_{1 \le s < t \le \ell} (t_{i_t} - t_{i_s}) \right] - a_0 \qquad (8) \\ &= \left[ \prod_{k=1}^{\ell} (t_{i_k} - t_j) - \prod_{k=1}^{\ell} t_{i_k} \right] \left[ \prod_{1 \le s < t \le \ell} (t_{i_t} - t_{i_s}) \right]. \end{aligned}$$

Since $\prod_{k=1}^{\ell} (t_{i_k} - t_j) > 0$, we have $a_1 t_j + a_2 t_j^2 + \cdots + a_\ell t_j^\ell > -a_0$. On the other hand, for each $1 \le k \le \ell$, we have

$$|t_{i_k} - t_j| = |i_k - j| \le m < t_{i_k}.$$

Thus, we conclude that $\prod_{k=1}^{\ell} (t_{i_k} - t_j) - \prod_{k=1}^{\ell} t_{i_k} < 0$, and $a_1 t_j + a_2 t_j^2 + \cdots + a_\ell t_j^\ell < a_0$. Therefore, we have Equation (7) for any $j \notin I$.

Next, we wish to show that the opposite of Equation (7) holds for every $j \in I - I_0$:

$$\left| a_1 t_j + a_2 t_j^2 + \cdots + a_\ell t_j^\ell \right| > a_0 \tag{9}$$

In fact, using Equation (8), it is sufficient to show that $\prod_{k=1}^{\ell} (t_{i_k} - t_j) < 0$, which is obviously true by the definition of $I$. In summary, we now have the following result:

$$\left| a_1 t_j + a_2 t_j^2 + \cdots + a_\ell t_j^\ell \right| \begin{cases} < a_0 & \text{if } j \notin I \\ = a_0 & \text{if } j \in I_0 \\ > a_0 & \text{if } j \in I - I_0 \end{cases}$$

Therefore, there exists $c > 0$, such that $v = \frac{1}{c}(a_1, \dots, a_\ell, 0, \dots, 0) \in \mathbb{R}^d$ satisfies the following condition:

$$|v \cdot q_j| \begin{cases} < 1 & \text{if } j \notin I \\ > 1 & \text{if } j \in I \end{cases}$$

This proves that $v \in \bigcap_{i \notin I} S_i - \bigcup_{i \in I} S_i$ and $I$ is a cell.

$\square$

For convenience, if $I = \bigcup_{k=1}^{\ell/2} [i_{2k-1}, i_{2k}]$, we say that $I$ is a cell *of order* $\ell/2$. In particular, $\emptyset$ is a cell of order 0. (Note that a cell may have multiple orders.)

### 3.3 The lower bounds

By now, we have developed all the tools to prove our main result. The idea of this proof is similar to that of Proposition 4. We wish to select as many good partitions as possible. Every subset in a good partition is a cell. Thus, $h_c(m, n, d)$ is at least the number of different good partitions.

**Proposition 11.** *If $n \geq 2$, $d \geq 2$, then*

$$h_c(m, n, d) \geq \begin{cases} n^{\Omega(m)} & \text{if } m \leq nd \\ g(2m, d)^{\Omega(n)} & \text{if } m \geq nd \end{cases} \tag{10}$$

*Proof.* First note that when $m \geq nd$, we have

$$\binom{2m}{d} \leq g(2m, d) = \binom{2m}{0} + \binom{2m}{1} + \cdots + \binom{2m}{d} \leq (d+1)\binom{2m}{d}.$$

Since $\binom{2m}{d} = (2m)(2m-1)\cdots(2m-d+1)/d! = (m/d)^{\Theta(d)}$, we have $g(2m, d) = (m/d)^{\Theta(d)}$ if $m \geq nd$. Therefore, if $m = \Theta(nd)$, it is easy to see that the two lower bounds for $h_c$ in Equation (10) coincide. Since $h_c(m, n, d)$ increases when $m$ or $d$ increase, it is sufficient to prove Equation (10) for even $d$. We distinguish several cases:

If $m \geq 2nd$, we calculate $s$, the number of good partitions of $[m]$ into $n$ subsets, each of which is a cell of order $d' = d/2$, and not of order less than $d'$:

$$\begin{aligned} s &\geq \binom{m - nd' - 1}{nd' - 1} \cdot (n-1)!^{d'} \\ &= \frac{(m - nd' - 1)\cdots(m - 2nd' + 1)}{(nd' - 1)!} \cdot (n-1)!^{d'} \\ &\geq \frac{(m - 2nd')^{nd'-1}}{(nd')^{nd'-1}} \cdot \left(\frac{n-1}{e}\right)^{(n-1)d'} \\ &= \left(\frac{m}{d}\right)^{\Theta(nd)}, \end{aligned}$$

where we use the inequality: $p! \geq (p/e)^p$ for any positive integer $p$.

If $2n \leq m \leq nd$, it is sufficient to consider the case when $m$ is a multiple of $2n$ because of the monotonicity of $h_c(m,n,d)$ in $m$. Suppose that $m = 2nk$ for some $1 \leq k \leq d/2$. We calculate $s$, the number of good partitions of $[m]$ into $n$ subsets, each of which is a cell of order $k$:

$$\begin{aligned} s &= \binom{nk}{k} \cdot \binom{(n-1)k}{k} \cdots \binom{k}{k} \\ &= \frac{(nk)!}{(k!)^n} \\ &\geq \left(\frac{nk}{e}\right)^{nk} / k^{nk} \\ &= n^{\Theta(m)}. \end{aligned}$$

If $m \leq n$, it is sufficient to consider the case when $m$ is even again because of the monotonicity of $h_c(m,n,d)$ in $m$. Denote $m' = m/2 \leq n/2$. We calculate $s$, the number of good partitions of $[m]$ into $n$ subsets, each of which is either empty or a cell of order 1:

$$\begin{aligned} s &= n \cdot (n-1) \cdots (n - m' + 1) \\ &\geq (n - m')^{m'} \\ &= n^{\Theta(m)}. \end{aligned}$$

If $n < m < 2n$, or $nd < m < 2nd$, Equation (10) still holds because of the monotonicity of $h_c(m,n,d)$ in $m$.

□

Combining the Proposition 11 with the two upper bounds that we got previously, we have the following corollary:

**Theorem 12.**

$$h_p(m,n,d),\ h_c(m,n,d) = \begin{cases} n^{\Theta(m)} & \textit{if } m = O(nd) \\ g(2m,d)^{\Theta(n)} & \textit{if } m = \Omega(nd) \end{cases} \tag{11}$$

Notice that when $m = \Theta(nd)$, the two expressions in Equation (11) coincide. In particular, consider the original problem that was proposed in the introduction. If $n = d$, and $m = O(n^2) = O(nd)$, we have $h_p(m,n,d) = n^{\Theta(m)}$ according to Equation (11). This shows that the upper bound $c^m$ (for some constant $c$) cannot be achieved in the worst case.

# 4 Conclusion

Our main problem arises in the proof of a computational lower bound on the complexity of volume algorithms. It was an open problem to derive a better upper bound for the ball partitioning problem in order to improve the computational lower bound from $\Omega(n^2/\log n)$ to $\Omega(n^2)$. However, the results derived in Section 3 show that this cannot be achieved. On the other hand, we derived a tight bound (up to constants in the exponent) for the ball partitioning and covering problems. It is still an open problem to tighten the gap on the complexity of volume computation.

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