Sections of convex bodies, statistical estimation and (in)stability

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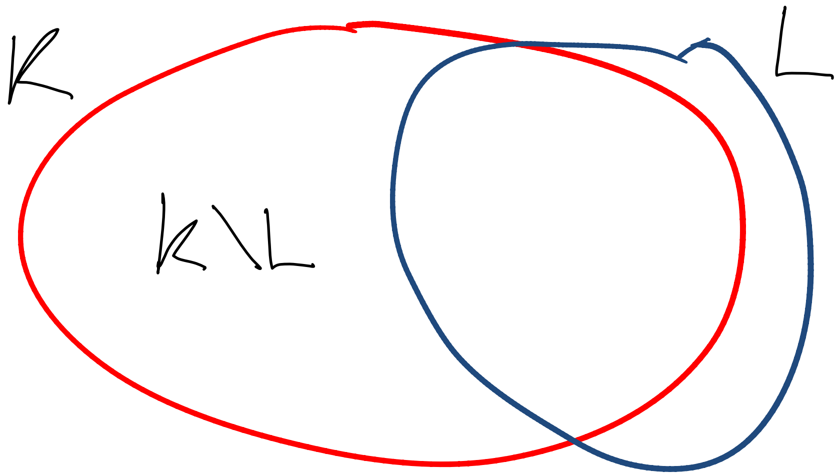
(joint work with Navin Goyal)

Minkowski's uniqueness theorem (injectivity)

- Theorem:
  If two centrally symmetric convex bodies have the same central sectional $(n-1)$-dimensional volumes, then they are equal.
- Stability?
- Upper bounds on error can be found in Groemer's book (inverse spherical Radon transform). Not useful for algorithms as they need too high precision.
- Focus here on distinguishing between polynomial and exponential dependence on dimension.

Distance



$$d(K,L) = \frac{\mathrm{vol}(K \setminus L)}{\mathrm{vol}\, K}$$

$$\mathrm{vol}\, K \geq \mathrm{vol}\, L$$

- $d(K,L)$ = total variation distance (statistical distance) between uniform distributions on K and L, f_K and f_L, respectively:

$$d(K,L) = \frac{1}{2}\int |f_K(x) - f_L(x)|\, dx$$

Stability of reconstruction

- Relative sectional volumes for $\theta \in S^{n-1}$:

$$A_K(\theta) := \frac{vol_{n-1}(K \cap \theta^{\perp})}{vol(K)}$$

- Can determine volume from it, so it is enough to determine centrally symmetric convex bodies (!)
- **Corollary** [Goyal R.]
  There exists universal constants $0 < c < 1, c' > 0$ such that for all n large enough:
  There exist two centrally symmetric convex bodies $K, L \subseteq R^n$ such that

$$\sup_{\theta} |A_K(\theta) - A_L(\theta)| < c^{\sqrt{n}}$$

  but $d_{TV}(K, L) > c'$.
- Relative v/s absolute not really important.

It is a corollary of

- **Theorem** [Goyal R ‘09]: There exists a distribution D on n-dimensional convex bodies and $c > 1$ such that if
  - f is a function that takes q points and outputs a convex body and,
  - when given q random points $X_1, \ldots, X_q$ from a convex body K according to D it satisfies

$$P\left(d_{TV}\left(K, f\left(X_1, \ldots, X_q\right)\right) < 1/8\right) \geq \frac{1}{2}.$$

Then $q > c^{\sqrt{n}}$

Idea: from random points to sections

- Approximate area of section by volume of narrow band around it.
- Given m random points from K, approximate relative volume of band by fraction of points in it.
- Concentration bound implies additive error for relative volume of band to within ϵ with $O\left(\frac{1}{\epsilon^2}\right)$ points *for a single band.*
- *How to get all bands simultaneously?*
- Want *uniform convergence* over all bands.

Connection between samples and relative sectional areas

- Use Vapnik Chervonenkis theory, “uniform convergence”.
- If $X = R^n$, $H =$ set of all halfspaces then VC-dimension d of (X, H) is $n + 1$.
- **Theorem** (Vapnik and Chervonenkis) (restatement of [Anthony and Bartlett, thm 4.3]):
  Suppose that H is a family of subsets of a set X and that (X, H) has finite VC-dimension d. Let D be a probability distribution on X. For any $0 < \epsilon < 1$ and $m > d$ a positive integer we have

$$P_{X_1,\ldots,X_m \sim D}\left((\forall A \in H) \left| P_D(A) - \frac{\#\{i: X_i \in A\}}{m} \right| < \epsilon \right) \geq 1 - 4m^{d+1} e^{-\epsilon^2 m/8}$$

Connection between samples and relative sectional areas

- **Corollary:**
  Let H be the set of all bands of the form $\{x \in R^n : v \cdot x \in [a, b]\}$ for any $a, b \in R \cup \{-\infty, \infty\}$ and $v \in R^n$. Let D be a probability distribution on R^n. For $0 < \epsilon < 1$ and $m > 10n$ a positive integer we have:

$$P_{X_1,\ldots,X_m \sim D}\left((\forall A \in H) \left| P_D(A) - \frac{\#\{i : X_i \in A\}}{m} \right| < \epsilon \right) \geq 1 - 4m^{10n} e^{-\epsilon^2 m/8}$$

- Proof: VC-dimension of bands is less than $10n$.

Connection between samples and relative sectional areas

Given a centrally symmetric convex body $K \subseteq \mathbb{R}^n$ and $\theta \in S^{n-1}$, let $A_K(\theta)$ denote the $(n-1)$-dimensional area of the section orthogonal to θ relative to $\mathrm{vol}(K)$:

$$A_K(\theta) := \frac{\mathrm{vol}_{n-1}(K \cap \theta^\perp)}{\mathrm{vol}\, K}.$$

Given $\delta > 0$, let $A_{K,\delta}$ be the following $\delta/2$-neighborhood approximation to A_K:

$$A_{K,\delta}(\theta) := \frac{1}{\delta} \frac{\mathrm{vol}(K \cap \{x : |\theta \cdot x| \le \delta/2\})}{\mathrm{vol}\, K}.$$

Given a random sample $X_1, \ldots, X_m$ from K, let $\tilde{A}_{K,\delta}$ be the following sample approximation to $A_{K,\delta}$ (and A_K):

$$\tilde{A}_{K,\delta}(\theta) := \frac{1}{\delta} \frac{\#\{i : |\theta \cdot X_i| \le \delta/2\}}{m}.$$

Connection between samples and relative sectional areas

For any direction θ, among all hyperplane sections of K perpendicular to θ, the one with maximal volume is the section containing the origin. This implies $A_{K,\delta}(\theta) \leq A_K(\theta)$ for all $\delta > 0$. If K is isotropic (covariance = identity), then it contains the ball of radius 1. This implies $K \cap (\theta^\perp + \delta\theta) \supseteq (1-\delta)K \cap \theta^\perp + \delta\theta$ and $\mathrm{vol}_{n-1}(K \cap (\theta^\perp + \delta\theta)) \geq (1-\delta)^{n-1}\,\mathrm{vol}_{n-1}(K \cap \theta^\perp)$. Thus,

$$A_{K,\delta}(\theta) \geq \left(1 - \frac{\delta}{2}\right)^{n-1} A_K(\theta) \geq \left(1 - \frac{(n-1)\delta}{2}\right) A_K(\theta).$$

If $\delta \leq 2\epsilon/(n-1)$ we have

$$(1-\epsilon)A_K(\theta) \leq A_{K,\delta}(\theta) \leq A_K(\theta).$$

Using that $B_n \subseteq K \subseteq c\sqrt{n}B_n$, we also have $c/\sqrt{n} \leq A_K(\theta) \leq 2n$. This implies the previous multiplicative bound on $A_{K,\delta}$ is also additive: If we let $\epsilon' = \epsilon/(2n)$ (so $\delta \leq \frac{\epsilon'}{n(n-1)}$) we have

$$A_K(\theta) - \epsilon' \leq A_{K,\delta}(\theta) \leq A_K(\theta). \tag{1}$$

Connection between samples and relative sectional areas

- **Corollary:**
  Let $K \subseteq R^n$ be a centrally symmetric isotropic convex body. Let $X_1, \ldots, X_m$ be iid and uniform in K. Let $0 < \epsilon'' < 1$. Then for $\delta = \epsilon''/(2n^2)$ and $m > 10n$ we have:
  $$P\left((\forall \theta \in S^{n-1})\left|\tilde{A}_{K,\delta}(\theta) - A_K(\theta)\right| < \epsilon''\right) \geq 1 - 4m^{10n}e^{-\epsilon''^4 m/(16n^2)}$$
- In particular, for $m = poly\left(n, \frac{1}{\epsilon''}, \log\frac{1}{\delta}\right)$ the rhs is at least $1 - \delta$.

Proof of the initial claim

- Corollary [Goyal R.]
  There exists universal constants $0 < c < 1, c' > 0$ such that for all n large enough:
  There exist two centrally symmetric convex bodies $K, L \subseteq R^n$ such that

  $$\sup_{\theta} |A_K(\theta) - A_L(\theta)| < c^{\sqrt{n}}$$

  but $d_{TV}(K, L) > c'$.
- Proof: Statistical lower bound in [G R'09] gives finite family of convex bodies such that any pair is at TV distance at least c' but to be estimated need more than $c_2^{\sqrt{n}}$ random points. At the same time VC corollary implies $c_2^{\sqrt{n}}$ points are enough to estimate $A_K(\theta)$ to within additive $c^{\sqrt{n}}$ for some $c > 0$. If the corollary were false, one could pinpoint the input body. This is a contradiction.

VC-dimension

- X: any set
- H: any family of subsets of X
- VC-dim of (X, H) is maximum $\#S$ over $S \subseteq X$ such that $\{S \cap A : A \in H\} = 2^S$.
- VC-dimension of halfspaces in R^n is $n + 1$ by Radon's theorem
  "Any set of $n + 2$ points in R^n can be partitioned into two subsets so that their convex hulls intersect".

Instability of volume

- Exact relative sectional areas determine volume (based on inverse spherical Radon transform).
- [Eldan] For some $\epsilon > 0$, need 2^{n^ϵ} random points to determine volume within factor 2^{n^ϵ} with probability 2^{-n^ϵ}.

$\Rightarrow$ As before then, volume needs relative sectional areas to within very high precision.

Instability of volume

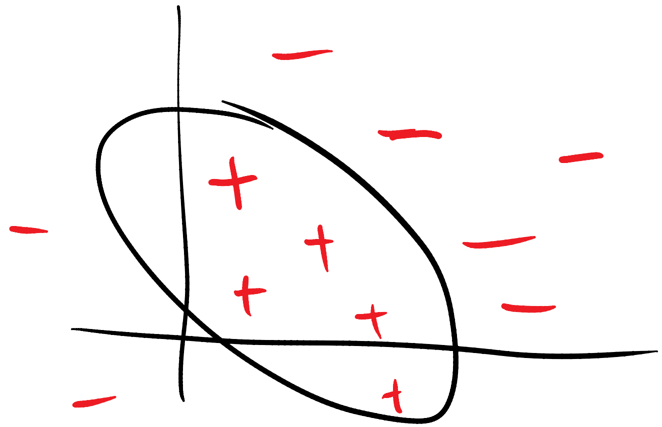
- **Corollary:** There exists a universal constant $\epsilon > 0$ such that for all n large enough: There exist two centrally symmetric convex bodies $K, L \subseteq R^n$ such that
$$\sup_{\theta} |A_K(\theta) - A_L(\theta)| < 2^{-n^{\epsilon}}$$
but $\mathrm{vol}(K)/\mathrm{vol}(L) > 2^{n^{\epsilon}}$

Message

- Instability of a functional (like volume) from random points implies instability from relative sectional areas.
- Argument is robust: instability from any other family with VC-dimension at most $poly(n)$.
- Also for Gaussian measure:

Similar result for Gaussian measure

- [Klivans O'Donnell Servedio] Need $2^{\Omega(\sqrt{n})}$ samples for reconstruction under Gaussian distribution (getting +,- labeled examples). (Algorithm with matching complexity.)



Questions

- Motivation: efficient (algorithmic) estimation of polytopes with $poly(n)$ facets (or vertices) from random points.
- (Q1) Injectivity of polytopes with $poly(n)$ (say, "n^2") facets (or vertices) from first "100" moment tensors.
  - [Gravin Lasserre Pasechnik Robins '11] Can recover a n-polytope with v vertices from $O(nv)$ moments along n random directions. But efficient estimation of higher order moments unlikely from samples.
  - [Frieze Jerrum Kannan '96] Can estimate parallelepiped from first 4 moment tensors
  - [Anderson Goyal R. '13] Can estimate any simplex from first 3 moment tensors.
- (Q2) Stability of reconstruction of polytopes with $poly(n)$ facets (or vertices) from central sectional areas. Algorithmic?