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ON THE MONOTONICITY OF THE EXPECTED VOLUME OF A RANDOM SIMPLEX

LUIS RADEMACHER

Abstract. Consider a random simplex in a d-dimensional convex body which is the convex hull of $d+1$ random points from the body. We study the following question: as a function of the convex body, is the expected volume of such a random simplex monotone non-decreasing under inclusion? We show that this is true when d is 1 or 2, but does not hold for $d \geq 4$. We also prove similar results for higher moments of the volume of a random simplex, in particular for the second moment, which corresponds to the determinant of the covariance matrix of the convex body. These questions are motivated by the slicing conjecture.

§1. *Introduction.* For a d-dimensional convex body K, let V_K denote the (random) volume of the convex hull of $d+1$ independent random points in K. In [**13**], Meckes asked whether for any pair of convex bodies $K, L \subseteq \mathbb{R}^d$,

$$K \subseteq L \quad \text{implies} \quad \mathbb{E}(V_K) \leq \mathbb{E}(V_L).$$

His "strong conjecture" claims that this is true. He also posed the following "weak conjecture": there exists a universal constant $c > 0$ such that

$$K \subseteq L \quad \text{implies} \quad \mathbb{E}(V_K) \leq c^d\, \mathbb{E}(V_L).$$

Clearly, the strong conjecture implies the weak conjecture. Meckes also wondered about natural generalizations to more than $d+1$ points, random polytopes, and higher moments. Later, Reitzner discussed the problem in [**18**] and [**19**, §2.2.1]; he asked whether $K \subseteq L$ implies

$$\mathbb{E}_{X_0,\ldots,X_n \in K}(\text{vol conv } X_0, \ldots, X_n) \leq \mathbb{E}_{X_0,\ldots,X_n \in L}(\text{vol conv } X_0, \ldots, X_n)$$

for arbitrary n.

While these are natural questions in understanding random polytopes, one of their motivations arises from their connection with the slicing conjecture (also known as the hyperplane conjecture or slicing problem), which says that all d-dimensional convex bodies of volume 1 have a hyperplane section of $(d-1)$-dimensional volume greater than or equal to a universal positive constant. Meckes's weak conjecture is equivalent to the slicing conjecture (see Appendix A). The slicing problem was apparently mentioned for the first time by Bourgain [**4**], and some equivalent formulations were discussed by Ball in [**2**]. Milman and Pajor [**15**] studied the problem systematically. It is

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one of the outstanding open problems in asymptotic convex geometry, and has attracted great interest because of its connections with some classical problems in convexity, such as the Busemann–Petty problem and Sylvester's problem [**7, 15**].

In this paper we show that Meckes's strong conjecture has a negative answer if $d \geq 4$ and a positive answer if d is 1 or 2. More precisely, we prove the following theorem.

THEOREM 1 (Random simplex). *If d is 1 or 2, and K and L are two d-dimensional convex bodies, then $K \subseteq L$ implies*

$$\mathbb{E}(V_K) \leq \mathbb{E}(V_L).$$

If $d \geq 4$, then there exist two convex bodies $K \subseteq L \subseteq \mathbb{R}^d$ such that

$$\mathbb{E}(V_K) > \mathbb{E}(V_L).$$

For the case of $d = 3$, numerical integration suggests that the same counterexample used for $d \geq 4$ should work for $d = 3$. However, the proof for higher d uses certain approximations for the integrals, which fail for $d = 3$; exact evaluation of the integrals might work but seems somewhat involved and is left as an open question.

From the proof of Theorem 1 one can infer the following counterexample: in d dimensions, let L be the convex hull of a half-ball (the unit ball with the constraint $x_1 \geq 0$, say) and a point at distance $\epsilon > 0$ from the center of the ball (the point $(-\epsilon, 0, \dots, 0)$, say). In other words, L is the union of a half-ball and a cone. Let K be L with the tip of the cone truncated at distance $\delta > 0$ (say, $K = L \cap \{x : x_1 \geq -\epsilon + \delta\}$). Then the proof of Theorem 1 shows that the pair K and L constitutes a counterexample to the monotonicity for $d \geq 4$ and ϵ, δ sufficiently small. Numerical integration suggests the same for $d = 3$.

The same counterexample and analysis work for higher moments and all dimensions greater than one (it is easy to see that for $d = 1$ the monotonicity holds for all moments). More precisely, we show the following.

THEOREM 2 (Higher moments). *If d is 2 or 3, then there exist an integer $k_0 \geq 1$ and two convex bodies $K \subseteq L \subseteq \mathbb{R}^d$ such that for any integer $k \geq k_0$ we have*

$$\mathbb{E}(V_K^k) > \mathbb{E}(V_L^k).$$

If $d \geq 4$, then there exist two convex bodies $K \subseteq L \subseteq \mathbb{R}^d$ such that for any integer $k \geq 1$ we have

$$\mathbb{E}(V_K^k) > \mathbb{E}(V_L^k).$$

The intuition for our answer to Meckes's question came from our solution to a simpler but related question posed by Vempala: is the determinant of the covariance matrix of a convex body monotone under inclusion? (The covariance matrix $A(\cdot)$ is defined in §2.) We answer Vempala's question by proving the following theorem.

THEOREM 3 (Determinant of covariance). *If d is 1 or 2 and K and L are two d-dimensional convex bodies, then $K \subseteq L$ implies* $\det A(K) \leq \det A(L)$. *If $d \geq 3$, then there exist two convex bodies $K \subseteq L \subseteq \mathbb{R}^d$ such that* $\det A(K) > \det A(L)$.

Vempala's question was also motivated by the slicing conjecture. As we shall show in Appendix A, the following weaker version of Theorem 3 is equivalent to the slicing conjecture: there exists a universal constant $c_1 > 0$ such that for any pair of convex bodies $K, M \subseteq \mathbb{R}^d$ we have

$$K \subseteq M \Longrightarrow \det A(K) \leq c_1^d \det A(M).$$

The high-level idea of the proof of Theorem 3 is as follows: to understand the monotonicity, it suffices to compute and understand the derivative of $\det A(\cdot)$ as one intersects the convex body with a moving half-space (Proposition 15). We then find conditions under which this derivative always has the right sign (Lemma 11 and the proof of Theorem 3). In the proof of Theorem 3 we show that understanding such a derivative is enough.

The following formula explains the connection between the determinant of the covariance matrix and the expected volume of a random simplex.

LEMMA 4. *Let K be a d-dimensional convex body, let $X_1, \ldots, X_d$ be random in K, and let $\mu(K) := \mathbb{E}X_1$ be the centroid of K. Then*

$$\begin{aligned}\det A(K) &= d!\, \mathbb{E}_{X_i \in K}((\operatorname{vol} \operatorname{conv} \mu(K), X_1, \ldots, X_d)^2)\\ &= \frac{d!}{d+1}\, \mathbb{E}(V_K^2).\end{aligned} \tag{1}$$

The first equality is known and easy to verify; see, for instance, [**7**, Proposition 1.3.3]. The second equality is a slight extension; see §3 for a proof.

In view of equation (1), one would think that if a pair of convex bodies provides an example that the monotonicity of $\det A(\cdot)$ does not hold, then it will also be such an example for the functional

$$K \mapsto \mathbb{E}(V_K).$$

Given these similarities, it should come as no surprise that techniques and examples similar to those used for $\det A(\cdot)$ also work for the expected volume of a random simplex and higher moments.

To prove Theorem 1, we use a special case of Crofton's theorem†; see [**21**, Ch. 5] or [**11**, Ch. 2]. Our special case is Proposition 16, which we prove here for completeness, partly because the proof of this version is elementary and partly because the formula was originally given as an informal statement rather than a theorem. Crofton's theorem has been formalized at least twice, once with differential geometry [**1**] and on another occasion with conditional probability [**6**]. It is likely that by using either of these two versions one could prove Theorem 1 in a simpler but less elementary way.

§2. *Preliminaries.* Let $K \subseteq \mathbb{R}^d$ be a convex body, and let $X_0, \ldots, X_d$ be random points in K. Let $\operatorname{vol}(\cdot)$ be the d-dimensional volume function and $\operatorname{vol}_k(\cdot)$ the k-dimensional volume function. Let V_K denote the random variable $V_K = \operatorname{vol}(\operatorname{conv}(X_0, \ldots, X_d))$. Suppose that X is random in K. Let $\mu(K)$ denote the

† Crofton's theorem is sometimes called Crofton's differential equation; it gives an expression for the derivative of a symmetric function of random points from a domain as the domain is perturbed.

centroid of K: $\mu(K) = \mathbb{E}(X)$. Let $A(K)$ be the covariance matrix of K:

$$A(K) = \mathbb{E}((X - \mu(K))(X - \mu(K))^T.$$

We say that K is isotropic if and only if $\mu(K) = 0$ and $A(K)$ is the identity matrix. It is easy to see that any convex body can be made isotropic by applying an affine transformation to it.

Given K and a hyperplane H, the *Steiner symmetrization* of K with respect to H is the convex body that results from the following process: for every line L orthogonal to H such that the segment $L \cap K$ is non-empty, shift the segment along L so that its midpoint lies in H. Similarly, given K and a half-space H, *Blaschke's shaking (Schüttelung)* of K with respect to H is the convex body that results from the following process: for every line L orthogonal to H such that the segment $L \cap K$ is non-empty, shift the segment along L so that one endpoint lies on the boundary of H while the whole segment stays inside H (see [**17**] for a discussion). If a given hyperplane H does not intersect the interior of K, then we define Blaschke's shaking of K with respect to H as the shaking defined before with respect to the half-space containing K and having H as boundary.

If we have a function f defined on an interval $[a, b]$, whenever we write the derivative of f at a we mean (implicitly) the one-sided derivative.

We will need the following known results.

THEOREM 5 (Blaschke [**20**, §8.2.3, Note 1]). *For any two-dimensional convex body K we have*

$$\frac{\mathbb{E}(V_K)}{\operatorname{vol}(K)} \leq \frac{1}{12},$$

with equality if and only if K is a triangle.

THEOREM 6 (Blaschke–Groemer [**20**, Theorem 8.6.3]). *Let $k \geq 1$ be an integer. Among all d-dimensional convex bodies,*

$$K \mapsto \frac{\mathbb{E}(V_K^k)}{\operatorname{vol}(K)^k}$$

is minimized if and only if K is an ellipsoid.

Let B_d be the d-dimensional unit ball and let S_{d-1} be the boundary of B_d. Let $\kappa_d := \operatorname{vol}(B_d) = \pi^{d/2}/\Gamma(1 + d/2)$ and $\omega_d := \operatorname{vol}_{d-1}(S_{d-1}) = d\kappa_d$.

THEOREM 7 (Random simplex in ball; see [**14**] or [**20**, Theorem 8.2.3]). *For any integer $k \geq 1$,*

$$\mathbb{E}(V_{B_d}^k) = \frac{1}{(d!)^k} \left(\frac{\kappa_{d+k}}{\kappa_d} \right)^{d+1} \frac{\kappa_{d(d+k+1)}}{\kappa_{(d+1)(d+k)}} \frac{\omega_1 \cdots \omega_k}{\omega_{d+1} \cdots \omega_{d+k}}.$$

THEOREM 8 (Simplex with origin in ball; see [**14**] or [**20**, Theorem 8.2.2]). *For any integer $k \geq 1$,*

$$\mathbb{E}_{X_i \in B_d}((\operatorname{vol}\operatorname{conv} 0, X_1, \ldots, X_d)^k) = \frac{1}{(d!)^k} \left(\frac{\kappa_{d+k}}{\kappa_d} \right)^d \frac{\omega_1 \cdots \omega_k}{\omega_{d+1} \cdots \omega_{d+k}}.$$

LEMMA 9 (Busemann random simplex inequality; see [5] or [20, Theorem 8.6.1]). *Among all d-dimensional convex bodies,*

$$K \mapsto \frac{\mathbb{E}_{X_i \in K}(\text{vol conv}(0, X_1, \ldots, X_d))}{\text{vol}(K)}$$

is minimized if and only if K is an ellipsoid centered at the origin. The minimum value is

$$\frac{1}{d!}\left(\frac{\kappa_{d+1}^{d}}{\kappa_{d}^{d+1}}\right)\frac{2}{\omega_{d+1}}.$$

LEMMA 10 (Ball volume ratio [3, p. 455]).

$$\sqrt{\frac{d}{2\pi}} \le \frac{\kappa_{d-1}}{\kappa_d} \le \sqrt{\frac{d+1}{2\pi}}.$$

Proof. The convexity of $\log \Gamma(x)$ implies that

$$(x+\alpha-1)^{\alpha} \le \frac{\Gamma(x+\alpha)}{\Gamma(x)} \le x^{\alpha} \quad \text{for } \alpha \in (0,1) \text{ and } x \ge 1.$$

The desired inequality follows. □

§3. *Proofs.*

3.1. *Proof of Theorem 3.* We will prove Theorem 3 first. Most of the work lies in proving the following dimension-dependent condition.

LEMMA 11. *Monotonicity under inclusion of $K \mapsto \det A(K)$ holds for some dimension d if and only if for any isotropic convex body $K \subseteq \mathbb{R}^d$ we have $\sqrt{d}B_d \subseteq K$.*

Proof. For the "if" part, suppose for a contradiction that $K, L \subseteq \mathbb{R}^d$ are two convex bodies for which the monotonicity does not hold, i.e. $K \subseteq L$ but

$$\det A(K) > \det A(L).$$

By the continuity of $\det A(\cdot)$ and the density of polytopes, we can assume without loss of generality that K is a polytope satisfying the same properties. Let m be the number of facets of K. Label the facets of K arbitrarily with labels $1, \ldots, m$. Let $F_i \subseteq \mathbb{R}^d$, $i = 0, \ldots, m$, be the following non-increasing sequence of convex bodies: $F_0 = L$, $F_m = K$, and $F_i = F_{i-1} \cap H_i$ where H_i is the unique half-space containing K and containing facet i of K in its boundary. Then there exists i such that the pair F_i, F_{i-1} is also a counterexample to the monotonicity. Let v be the unit outer normal to facet i of K. Consider the path from F_{i-1} to F_i induced by pushing H_i in; formally, the path is given by

$$F(t) = F_{i-1} \cap H(t) \quad \text{for } t \in [a, b],$$

where $H(t) = \{x \in \mathbb{R}^d : v \cdot x \le t\}$, $a = \sup_{x \in F_{i-1}} v \cdot x$ and $b = \sup_{x \in F_i} v \cdot x$. The function $t \mapsto \det A(F(t))$ is continuous in $[a, b]$ and differentiable in (a, b), and the lack of monotonicity implies that there exists $\bar{t} \in (a, b)$ such that its derivative is positive at $\bar{t}$. Now, even though this derivative is not invariant under non-singular affine transformations, its sign *is* invariant. Thus, we can assume without loss of generality that $F_{\bar{t}}$ is in isotropic position, and Proposition 15 then tells us that

$$\mathbb{E}_{X \in S_{\bar{t}}}(\|X\|^2) < d$$

where $S_{\bar{t}} = F_{\bar{t}} \cap \operatorname{bdry} H(\bar{t})$. In particular, there exists $x \in S_{\bar{t}}$ such that $\|x\| < \sqrt{d}$, which implies $\sqrt{d} B_d \nsubseteq F_{\bar{t}}$.

For the "only if" part, suppose that there is an isotropic convex body $K \subseteq \mathbb{R}^d$ and a point $x \in \operatorname{bdry} K$ such that $\|x\| < \sqrt{d}$. By continuity and an approximation argument we can, without loss of generality, replace K and x so that x is an extreme point of K while still satisfying $\|x\| < \sqrt{d}$ and isotropy†. Let $v \in \mathbb{R}^d \setminus \{0\}$ and $a < 0$ determine a half-space $H = \{x \in \mathbb{R}^n : v \cdot x \geq a\}$ containing K whose boundary intersects K only at x. Let $H_t = \{x \in \mathbb{R}^n : v \cdot x \geq t\}$. Let L_t be the convex body $K \cap H_t$. Then by continuity, Proposition 15 and the fact that $\|x\| < d$, we have that there exists $\epsilon > 0$ such that for all $t \in (a, a + \epsilon)$,

$$\frac{d}{dt} \det A(L_t) > 0.$$

This implies that $\det A(K) < \det A(L_{a+\epsilon})$ while $L_{a+\epsilon} \subseteq K$. □

Proof of Theorem 3. The assertions follow immediately from Lemma 11 and the fact that any d-dimensional isotropic convex body contains the ball of radius $\sqrt{(d+2)/d}$ centered at the origin and that this is best possible; see [**15, 22**] and [**10**, Theorem 4.1]. □

3.2. *Proof of Theorems 1 and 2.* We will now prove Theorems 1 and 2. We begin with a dimension-dependent condition similar to that in Lemma 11.

LEMMA 12. *For a given integer $k \geq 1$ and dimension d, monotonicity under inclusion of*

$$K \mapsto \mathbb{E}(V_K^k)$$

holds as K ranges over d-dimensional convex bodies if and only if for any convex body $K \subseteq \mathbb{R}^d$, any $x \in \operatorname{bdry} K$ and any $X_1, \ldots, X_d$ random in K we have

$$\mathbb{E}(V_K^k) \leq \mathbb{E}((\operatorname{vol} \operatorname{conv} x, X_1, \ldots, X_d)^k). \tag{2}$$

Proof. The proof is essentially the same as the proof of Lemma 11, with Proposition 15 replaced by Proposition 16, $q = d + 1$ and

$$f(x_0, \ldots, x_d) = (\operatorname{vol} \operatorname{conv} x_0, \ldots, x_d)^k,$$

and without using isotropy. □

Next, we verify the dimension-dependent condition for $k = 1$ in $\mathbb{R}^2$ by means of the following lemma (which gives a lower bound on the right-hand side of (2)) and Blaschke's maximality of the triangle for Sylvester's problem, i.e. Theorem 5 (which gives an upper bound for the left-hand side).

† For example, add a point $x_\alpha = \alpha x$ for $\alpha > 1$ and take the convex hull between K and x_α to get a convex body K_α. We have that x_α is an extreme point of K_α. For some α sufficiently close to 1 and $T_\alpha = A(K_\alpha)^{-1/2}$, we have that $T_\alpha K_\alpha$ is isotropic and $\|T_\alpha x_\alpha\| < d$.

LEMMA 13. *Let $K \subseteq \mathbb{R}^2$ be a convex body and let $x \in \operatorname{bdry} K$. Then*

$$\frac{\mathbb{E}_{X_1, X_2 \in K}(\operatorname{vol} \operatorname{conv} x, X_1, X_2)}{\operatorname{vol} K} \geq \frac{8}{9\pi^2}. \tag{3}$$

Proof. The continuity and affine-invariance of the left-hand side of (3) and a standard compactness argument imply existence of a minimum K and x.

To prove the inequality, we will show with a series of symmetrizations that half of a ball centered at x minimizes the left-hand side. The intuition needed to understand the effect of Steiner symmetrization and Blaschke's shaking (see §2 for a brief review) is the following [**17**, §3]: if one picks three points at random from three vertical segments in the plane that are allowed to move vertically (one point from each segment), then the expected area of the convex hull of those three points is a strictly increasing function of the area of the triangle formed by the midpoints of the segments. This implies, for instance, that Steiner symmetrization decreases the expected area of a random triangle: the area of the triangle formed by the midpoints is zero when the midpoints lie on a common line.

Here is the sequence of symmetrizations.

(1) *Steiner symmetrization.* Let L be any supporting line of K through x. Let $L^{\perp}$ be a line orthogonal to L through x. Let $\bar{K}$ be the Steiner symmetrization of K with respect to $L^{\perp}$. If $\bar{K} \neq K$, then $\bar{K}$ has a strictly smaller value than K in the left-hand side of (3); see [**9**, Lemma 4] or [**5**].

(2) *Blaschke's shaking (Schüttelung) with respect to L.* Take each chord of $\bar{K}$ perpendicular to L and shift it in the direction orthogonal to L so that its endpoint nearest to L lies on L. The union of the shifted chords is a convex body that we denote by $\bar{\bar{K}}$. Lemma 14 shows that this operation can only decrease the value of the left-hand side of (3), so we now know that the set of pairs that are invariant under the previous step and this step contains a minimizer. Denote by $\mathcal{S}$ the family of pairs satisfying such an invariance.

(3) The left-hand side of (3) just halves if one replaces $\bar{\bar{K}}$ by its symmetrization around x, and this symmetrization is a centrally symmetric convex body given the previous two steps. Thus,

$$\inf_{(K,x) \in \mathcal{S}} \frac{\mathbb{E}_K(\operatorname{vol} \operatorname{conv} x, X_1, X_2)}{\operatorname{vol} K} \geq 2 \inf_{K'} \frac{\mathbb{E}_{K'}(\operatorname{vol} \operatorname{conv} 0, X_1, X_2)}{\operatorname{vol} K'}, \tag{4}$$

where K' ranges over all centrally symmetric convex bodies. Lemma 9 implies that ellipses are the only minimizers of the right-hand side of (4) and, as half of an ellipse around the origin together with the origin form a pair in $\mathcal{S}$, we conclude that half of a disk centered at x is a minimizer.

To get the right-hand side in (3), we just need to evaluate the left-hand side for $x = 0$ and K being one half of the unit disk. For the numerator, the symmetry of the problem implies that the average for a half-disk and the origin is the same as the average for the disk and the origin. Thus, Theorem 8 implies that

$$\mathbb{E}(\operatorname{vol} \operatorname{conv} x, X_1, X_2) = \frac{4}{9\pi},$$

while the denominator in (3) is the area of a half-disk, $\pi/2$. $\square$

We believe that half of an ellipse centered at x is the only kind of minimizer possible.

Proof of Theorem 1. For the first part ($d \le 2$), the assertion is clearly true for $d = 1$. For $d = 2$, Theorem 5 (Blaschke's maximality of the triangle for Sylvester's problem) and Lemmas 12 and 13 imply the desired conclusion.

The second part ($d \ge 4$) is a special case of Theorem 2. □

Numerical experiments suggest that a simplex and the center point of a facet should work as a counterexample for the monotonicity as in Theorem 1 in $\mathbb{R}^3$, and it should work in higher dimensions. Similar numerical experiments indicate that half of the unit ball and the origin also constitute a counterexample in $\mathbb{R}^3$.

Proof of Theorem 2. Let K be the half-ball with $x_d \ge 0$.

For L being the ball of volume $\mathrm{vol}(K)$, Theorem 6 implies that

$$\mathbb{E}(V_K^k) \ge \mathbb{E}(V_L^k).$$

Theorem 7 implies that

$$\begin{aligned}\mathbb{E}(V_L^k) &= \frac{1}{2^k}\,\mathbb{E}(V_{B_d}^k)\\ &= \frac{1}{2^k (d!)^k}\left(\frac{\kappa_{d+k}}{\kappa_d}\right)^{d+1}\frac{\kappa_{d(d+k+1)}}{\kappa_{(d+1)(d+k)}}\,\frac{\omega_1\cdots\omega_k}{\omega_{d+1}\cdots\omega_{d+k}}.\end{aligned}$$

On the other hand, symmetry and Theorem 8 imply that

$$\begin{aligned}\mathbb{E}_{X_i\in K}((\mathrm{vol\,conv}\,0, X_1,\dots,X_d)^k) &= \mathbb{E}_{X_i\in B_d}((\mathrm{vol\,conv}\,0, X_1,\dots,X_d)^k)\\ &= \frac{1}{(d!)^k}\left(\frac{\kappa_{d+k}}{\kappa_d}\right)^{d}\frac{\omega_1\cdots\omega_k}{\omega_{d+1}\cdots\omega_{d+k}}.\end{aligned}$$

Upon combining the previous claims, we get

$$\frac{\mathbb{E}_{X_i\in K}((\mathrm{vol\,conv}\,0, X_1,\dots,X_d)^k)}{\mathbb{E}(V_K^k)} \le 2^k\,\frac{\kappa_d}{\kappa_{d+k}}\,\frac{\kappa_{(d+1)(d+k)}}{\kappa_{d(d+k+1)}}. \tag{5}$$

When d is 2 or 3, a tedious but straightforward use of Stirling's formula shows that (5) goes to 0 as k tends to infinity. Lemma 12 completes the argument in this case.

If $d \ge 4$, Lemma 10 and inequality (5) give

$$\begin{aligned}&\frac{\mathbb{E}_{X_i\in K}((\mathrm{vol\,conv}\,0, X_1,\dots,X_d)^k)}{\mathbb{E}(V_K^k)}\\ &\quad\le 2^k\left(\frac{(d+2)\cdots(d+k+1)}{(d(d+k+1)+1)\cdots(d(d+k+1)+k)}\right)^{1/2}\\ &\quad\le 2^k\left(\frac{d+k+1}{d(d+k+1)+k}\right)^{k/2}\end{aligned}$$

(where we have used the inequality $a/b \le (a+1)/(b+1)$ whenever $0 \le a \le b$), and this is less than 1 for any $k \ge 1$. Lemma 12 completes the argument. □

LEMMA 14. *Let $K \subseteq \mathbb{R}^2$ be a convex body, let $x \in$ bdry K, and let L be a supporting line of K at x. Assume, in addition, that K is symmetric about the line through x orthogonal to L. Let $\bar{K}$ be Blaschke's shaking of K with respect to L. Then*

$$\mathbb{E}_{X_i \in K}(\text{vol conv } x, X_1, X_2) \geq \mathbb{E}_{X_i \in \bar{K}}(\text{vol conv } x, X_1, X_2). \tag{6}$$

Proof. Without loss of generality, translate and rotate everything so that x is at the origin and L is the "x" axis. Let $t > 0$ be half of the width of K along the x axis. For any $u \in [-t, t]$, define functions $\alpha(u)$ and $l(u)$ so that the vertical chords of K have the form $\{u\} \times [\alpha(u), \alpha(u) + l(u)]$ (i.e. α is the "bottom" of the chord and l is its length). We have

$$\begin{aligned}
&\mathbb{E}_{X_i \in K}(\text{vol conv } 0, X_1, X_2) \\
&\quad = \frac{1}{2(\text{vol } K)^2} \int_{-t}^{t} \int_{0}^{t} \int_{0}^{l_1(u_1)} \int_{0}^{l_2(u_2)} \left|\det\begin{pmatrix} u_1 & \alpha(u_1) + v_1 \\ u_2 & \alpha(u_2) + v_2 \end{pmatrix}\right| \\
&\qquad + \left|\det\begin{pmatrix} -u_1 & \alpha(u_1) + v_1 \\ u_2 & \alpha(u_2) + v_2 \end{pmatrix}\right| dv_2\, dv_1\, du_2\, du_1.
\end{aligned}$$

Let $f(\alpha_1, \alpha_2)$ denote the integrand for fixed values of the integration variables:

$$f(\alpha_1, \alpha_2) = \left|\det\begin{pmatrix} u_1 & \alpha_1 + v_1 \\ u_2 & \alpha_2 + v_2 \end{pmatrix}\right| + \left|\det\begin{pmatrix} -u_1 & \alpha_1 + v_1 \\ u_2 & \alpha_2 + v_2 \end{pmatrix}\right|.$$

The function f is clearly convex. Moreover,

$$\begin{aligned}
f(\alpha_1, \alpha_2) = f(-\alpha_1 - 2v_1, \alpha_2) &= f(\alpha_1, -\alpha_2 - 2v_2) \\
&= f(-\alpha_1 - 2v_1, -\alpha_2 - 2v_2).
\end{aligned}$$

So, for $\lambda_i = \alpha_i / 2(\alpha_i + v_i)$ and by using convexity, we have

$$\begin{aligned}
&f(\alpha_1, \alpha_2) \\
&\quad = (1 - \lambda_1)(1 - \lambda_2) f(\alpha_1, \alpha_2) + (1 - \lambda_1)\lambda_2 f(\alpha_1, -\alpha_2 - 2v_2) \\
&\qquad + \lambda_1(1 - \lambda_2) f(-\alpha_1 - 2v_1, \alpha_2) \\
&\qquad + \lambda_1\lambda_2 f(-\alpha_1 - 2v_1, -\alpha_2 - 2v_2) \\
&\quad \geq f(0, 0).
\end{aligned}$$

Plugging this into our integral gives

$$\begin{aligned}
&\mathbb{E}_{X_i \in K}(\text{vol conv } 0, X_1, X_2) \\
&\quad \geq \frac{1}{2(\text{vol } K)^2} \int_{-t}^{t} \int_{0}^{t} \int_{0}^{l_1(u_1)} \int_{0}^{l_2(u_2)} \left|\det\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\right| \\
&\qquad + \left|\det\begin{pmatrix} -u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\right| dv_2\, dv_1\, du_2\, du_1 \\
&\quad = \mathbb{E}_{X_i \in \bar{K}}(\text{vol conv } 0, X_1, X_2).
\end{aligned}$$

□

3.3. *Crofton's formula and relatives.*

PROPOSITION 15 (Derivative of $\det A(K)$). *Let $K \subseteq \mathbb{R}^d$ be an isotropic convex body. Let $v \in \mathbb{R}^d$ be a unit vector. Let $a = \inf_{x \in K} v \cdot x$, $b = \sup_{x \in K} v \cdot x$*

and $H_t = \{x \in \mathbb{R}^d : v \cdot x \geq t\}$*. Let* $K_t = K \cap H_t$ *and* $S_t = K \cap \operatorname{bdry} H_t$*. Then*

$$\frac{d}{dt} \det A(K_t)\bigg|_{t=a} = (d - \mathbb{E}_{X\in S_a}(\|X\|^2))\frac{\operatorname{vol}_{d-1} S_a}{\operatorname{vol} K}.$$

Proof. We have

$$\begin{aligned} A(K_t) &= \mathbb{E}_{X\in K_t}((X-\mu(K_t))(X-\mu(K_t))^T) \\ &= \mathbb{E}_{X\in K_t}(XX^T) - \mu(K_t)\mu(K_t)^T. \end{aligned}$$

By isotropy, $\mu(K) = 0$, and this implies that

$$\frac{d}{dt} A(K_t)\bigg|_{t=a} = \frac{d}{dt}\, \mathbb{E}_{X\in K_t}(XX^T)\bigg|_{t=a}. \tag{7}$$

We then use the identity

$$\frac{d}{dM} \det M = (M^{-1})^T \det M$$

to deduce that

$$\begin{aligned} \frac{d}{dt} \det A(K_t) &= \frac{d}{dM} \det M\bigg|_{M=A(K_t)} \cdot \frac{d}{dt} A(K_t) \\ &= \det(A(K_t))(A(K_t)^{-1})^T \cdot \frac{d}{dt} A(K_t), \end{aligned}$$

where the dot "$\cdot$" represents the Frobenius inner product of matrices, $M \cdot N = \sum_{ij} M_{ij} N_{ij}$. This, together with isotropy and (7), gives

$$\begin{aligned} \frac{d}{dt} \det A(K_t)\bigg|_{t=a} &= I \cdot \frac{d}{dt}\, \mathbb{E}_{X\in K_t}(XX^T)\bigg|_{t=a} \\ &= \frac{d}{dt}\, \mathbb{E}_{X\in K_t}(\|X\|^2)\bigg|_{t=a}. \end{aligned}$$

To conclude, evaluate the following at $t = a$, using isotropy in the second step:

$$\begin{aligned} \frac{d}{dt}\, \mathbb{E}_{X\in K_t}(\|X\|^2) &= \frac{d}{dt} \frac{1}{\operatorname{vol} K_t} \int_t^b \mathbb{E}_{X\in S_\alpha}(\|X\|^2) \operatorname{vol}_{d-1}(S_\alpha)\, d\alpha \\ &= \frac{\operatorname{vol}_{d-1}(S_t)}{\operatorname{vol} K_t}(\mathbb{E}_{X\in K_t}(\|X\|^2) - \mathbb{E}_{X\in S_t}(\|X\|^2)). \qquad \square \end{aligned}$$

We say that $f : U^q \to V$ is symmetric if and only if for any permutation π of $\{1, \dots, q\}$ and any $x \in U^q$ we have $f(x) = f(x_{\pi(1)}, \dots, x_{\pi(q)})$.

PROPOSITION 16 (General derivative, Crofton). *Let* $K \subseteq \mathbb{R}^d$ *be a convex body. Let* $v \in \mathbb{R}^d$ *be a unit vector. Let* $a = \inf_{x\in K} v \cdot x$ *and* $b = \sup_{x\in K} v \cdot x$*, and let* $H_t = \{x \in \mathbb{R}^d : v \cdot x \geq t\}$*. Let* $K_t = K \cap H_t$ *and* $S_t = K \cap \operatorname{bdry} H_t$*. Let* $f : (\mathbb{R}^d)^q \to \mathbb{R}$ *be a symmetric continuous function. Let* $X_1, \dots, X_q$ *be independent random points in* K*. Then*

$$\begin{aligned} &\frac{d}{dt}\, \mathbb{E} f(X_1, \dots, X_q)\bigg|_{t=a} \\ &\quad = q(\mathbb{E} f(X_1, \dots, X_q) - \mathbb{E}(f(X_1, \dots, X_q) \mid X_1 \in S_a))\frac{\operatorname{vol}_{d-1} S_a}{\operatorname{vol} K}. \end{aligned}$$

(A slightly different proof should work under the weaker assumption that f is bounded and measurable but not necessarily continuous.)

Proof. Use repeatedly the identity

$$\frac{d}{dt}\int_t^b u(x,t)\,dx = -u(t,t) + \int_t^b \frac{d}{dt}u(x,t)\,dx$$

and the symmetry of f to get

$$\begin{aligned}
&\frac{d}{dt}\int_{K_t^q} f(x)\,dx\bigg|_{t=a} \\
&\quad = \frac{d}{dt}\int_{[t,b]^q}\int_{S_{\alpha_1}\times\cdots\times S_{\alpha_q}} f(x)\,dx\,d\alpha\bigg|_{t=a} \\
&\quad = -q\int_{[a,b]^{q-1}}\int_{S_a}\int_{S_{\alpha_2}\times\cdots\times S_{\alpha_q}} f(x)\,dx_q\cdots dx_2\,dx_1\,d\alpha_q\cdots d\alpha_2. \qquad (8)
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{d}{dt}\,\mathbb{E}_{X_i\in K_t}(f(X_1,\dots,X_q))\bigg|_{t=a} \\
&\quad = \frac{d}{dt}\frac{1}{(\operatorname{vol} K_t)^q}\int_{K_t^q} f(x)\,dx\bigg|_{t=a} \\
&\quad = \frac{1}{(\operatorname{vol} K_t)^{2q}}\biggl((\operatorname{vol} K_t)^q\biggl[\frac{d}{dt}\int_{K_t^q} f(x)\,dx\biggr] \\
&\qquad\qquad + q(\operatorname{vol} K_t)^{q-1}\operatorname{vol}_{d-1}(S_t)\int_{K_t^q} f(x)\,dx\biggr)\bigg|_{t=a} \\
&\quad = \frac{\operatorname{vol}_{d-1}(S_a)}{\operatorname{vol} K}\biggl(\frac{1}{(\operatorname{vol} K)^{q-1}\operatorname{vol}_{d-1}(S_a)}\biggl[\frac{d}{dt}\int_{K_t^q} f(x)\,dx\biggr]_{t=a} \\
&\qquad\qquad + \frac{q}{(\operatorname{vol} K)^q}\int_{K^q} f(x)\,dx\biggr).
\end{aligned}$$

To conclude, use (8) and interpret the integrals as expectations. $\square$

3.4. *Proof of Lemma 4.* If Y is a random d-dimensional vector with second moments and $Y_1,\dots,Y_d$ are identically distributed independent copies of Y, then the following identity is known and easy to verify by expanding the determinant:

$$\det \mathbb{E}YY^T = \frac{1}{d!}\,\mathbb{E}((\det Y_1,\dots,Y_d)^2). \qquad (9)$$

The first identity in the lemma follows immediately from this. To get the second identity, i.e. equation (1), let X_0 be random in K and consider

$$\begin{aligned}
V_K &= \frac{1}{d!}\,|\det(X_1-X_0,\dots,X_d-X_0)| \\
&= \frac{1}{d!}\left|\det\begin{pmatrix} X_0 & \cdots & X_d \\ 1 & \cdots & 1\end{pmatrix}\right|.
\end{aligned}$$

Taking the expectation of the squares and using equation (9) gives

$$\mathbb{E}(V_K^2) = \frac{d+1}{d!} \det\begin{pmatrix} \mathbb{E}XX^T & \mu(K) \\ \mu(K)^T & 1 \end{pmatrix}.$$

The left-hand side is invariant under translation of K, so the right-hand side must be as well, and without loss of generality we can assume $\mu(K) = 0$. Equation (1) follows.

§4. *Discussion.* Finally, we list a few open questions related to this work.

(1) *Random polytopes.* As mentioned in the introduction, Meckes and Reitzner asked about monotonicity of the expected volume of a random polytope with n vertices, not just a random simplex as in this paper. It is easy to see that given d-dimensional convex bodies K and L with $K \subset L$ there exists $n_0 = n_0(K, L)$ such that for $n \geq n_0$ we have

$$\mathbb{E}_{X_0,\dots,X_n \in K} \operatorname{vol} \operatorname{conv} X_0, \dots, X_n \leq \mathbb{E}_{X_0,\dots,X_n \in L} \operatorname{vol} \operatorname{conv} X_0, \dots, X_n.$$

Can one choose n_0 so that it may depend on d but is independent of K and L?

(2) *Three-dimensional case.* For Meckes's strong conjecture, find an easy argument to disprove it for $d = 3$.

(3) *Slicing conjecture.* Understand Meckes's weak conjecture.

(4) *Sylvester's problem.* Show that among all d-dimensional convex bodies,

$$K \mapsto \frac{\mathbb{E}_{X_i \in K}(\operatorname{vol} \operatorname{conv}(X_0, \dots, X_d))}{\operatorname{vol}(K)}$$

is maximized if K is a simplex. (This is known to imply the slicing conjecture [7].)

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A. *Appendix.* For completeness, we prove here the equivalence between the slicing conjecture, Meckes's weak conjecture and Vempala's question. Slight variations of the following argument have been put forth by Meckes (personal communication) and, later, independently by Vempala and Dadush (personal communication). The main ingredients are Klartag's [12] answer to the isomorphic slicing problem and a Khinchine-type inequality (reverse Hölder inequality) [16, Appendix III].

It is known (see [2] or [7, §1.5]) that the slicing conjecture as stated in the introduction (in terms of hyperplane sections) is equivalent to the existence of a universal upper bound on the isotropic constant of a convex body, which is defined as follows: given a convex body $K \subseteq \mathbb{R}^d$, the isotropic constant L_K of K is given by

$$L_K^{2d} = \frac{\det A(K)}{(\operatorname{vol} K)^2}. \tag{A1}$$

CONJECTURE A.1 (Slicing conjecture). *There exists a universal constant* $c_3 > 0$ *such that for any* d *and any convex body* $K \subseteq \mathbb{R}^d$ *we have* $L_K \le c_3$.

We now state Klartag's result from [12]. For a pair of convex bodies $K, M \subseteq \mathbb{R}^d$, define the Banach–Mazur distance to be

$$d_{\mathrm{BM}}(K, M) := \inf \big\{ a \ge 1 : K \subseteq T(M) \subseteq aK, \\ T : \mathbb{R}^d \to \mathbb{R}^d \text{ is a non-singular affine transformation} \big\}.$$

THEOREM A.2 (Isomorphic slicing problem [12]). *There exists* $c > 0$ *such that if* $K \subseteq \mathbb{R}^d$ *is a convex body and* $\epsilon > 0$*, then there is a convex body* $M \subseteq \mathbb{R}^d$ *such that:*

- $d_{\mathrm{BM}}(K, M) < 1 + \epsilon$;
- $L_M < c/\sqrt{\epsilon}$.

Here is the Khinchine-type inequality that we need.

LEMMA A.3 ([16, Appendix III], [7, §2.1] or [8, p. 717]). *There exists a constant* $c > 0$ *such that if* $f : \mathbb{R}^d \to \mathbb{R}^+$ *is a semi-norm,* $K \subseteq \mathbb{R}^d$ *is a convex body and* $1 \le p < \infty$*, then*

$$\frac{1}{\mathrm{vol}\, K} \int_K f(x)\, dx \le \left(\frac{1}{\mathrm{vol}\, K} \int_K f(x)^p\, dx \right)^{1/p} \le \frac{cp}{\mathrm{vol}\, K} \int_K f(x)\, dx.$$

The following proposition states the desired equivalences between the slicing conjecture and the monotonicity questions.

PROPOSITION A.4. *For any* $1 \le p < \infty$*, the following statements are equivalent.*

(1) *(Vempala's question.) There exists* $c_1 > 0$ *such that for any pair of convex bodies* $K, M \subseteq \mathbb{R}^d$ *we have*

$$K \subseteq M \Longrightarrow \det A(K) \le c_1^d \det A(M).$$

(2) *(Meckes's weak conjecture for the pth moment.) There exists* $c_2 > 0$ *such that for any pair of convex bodies* $K, M \subseteq \mathbb{R}^d$ *we have*

$$K \subseteq M \Longrightarrow \mathbb{E}(V_K^p) \le c_2^d\, \mathbb{E}(V_M^p).$$

(3) *(The slicing conjecture.) Conjecture A.1.*

Proof. (3) $\Longrightarrow$ (1): Let $K, M \subseteq \mathbb{R}^d$ be convex bodies such that $K \subseteq M$. Then, by using equations (1) and (A1), we have

$$\frac{\mathbb{E}V_K^2}{(\mathrm{vol}\, K)^2} = \frac{d+1}{d!} \frac{\det A(K)}{(\mathrm{vol}\, K)^2} = \frac{d+1}{d!} L_K^{2d}, \tag{A2}$$

and a similar equality can be obtained for M.

It is known that the isotropic constant has a universal lower bound $c > 0$ over all dimensions and all convex bodies [2, 4, 15]. This, together with our assumption, implies that $c \leq L_M, L_K \leq c_3$; that is (using equation (A2)),

$$\frac{\mathbb{E}V_K^2}{(\mathrm{vol}\ K)^2} \leq \frac{d+1}{d!}\, c_3^{2d}$$

and

$$\frac{d+1}{d!}\, c^{2d} \leq \frac{\mathbb{E}V_M^2}{(\mathrm{vol}\ M)^2}.$$

In conjunction with the fact that vol $K \leq$ vol M, this gives

$$\mathbb{E}(V_K^2) \leq \left(\frac{c_3}{c}\right)^{2d} \mathbb{E}(V_M^2).$$

(1) $\Longrightarrow$ (3): The positive solution to the isomorphic slicing problem (Theorem A.2) with $\epsilon = 1$ implies that there exists a constant $c > 0$ such that for any convex body $K \subseteq \mathbb{R}^d$ there exists another convex body $M \subseteq \mathbb{R}^d$ satisfying $d_{\mathrm{BM}}(K, M) \leq 2$ and $L_M \leq c$. Consider an arbitrary convex body $K \subseteq \mathbb{R}^d$ and let M be the convex body given by Theorem A.2. As the isotropic constants L_K, L_M and $\det A(\cdot)$ are invariant under affine transformations, we can assume without loss of generality that $K \subseteq M \subseteq 2K$. This and (A1) imply that

$$L_K^{2d} = \frac{\det A(K)}{(\mathrm{vol}\ K)^2} \leq 2^{2d}\, \frac{c_1^d \det A(M)}{(\mathrm{vol}\ M)^2} \leq (2c\sqrt{c_1})^{2d}.$$

(1) $\Longleftrightarrow$ (2): This is an easy consequence of our Khinchine-type inequality (Lemma A.3) and equation (A2): iterated use of Lemma A.3 implies that

$$\mathbb{E}V_K \leq (\mathbb{E}(V_K^p))^{1/p} \leq c^{d+1} p^{d+1}\, \mathbb{E}V_K,$$

and the claimed equivalence follows. □

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Luis Rademacher,
School of Computer Science and Engineering,
The Ohio State University,
Columbus, OH 43210,
U.S.A.
E-mail: lrademac@cse.ohio-state.edu