Computing Equilibrium Prices in Exchange Economies with Tax Distortions

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Abstract

We consider the computation of equilibrium prices in market settings where purchases of goods are subject to taxation. While this scenario is the standard one in applied computational work, so far it has not been an object of study in theoretical computer science. Taxes introduce significant distortions: equilibria are no longer Pareto optimal, sufficient conditions for uniqueness do not continue to guarantee it, existence itself must be revisited. We analyze the effects of these distortions on scenarios which, in the absence of taxes, admit polynomial time algorithms, and prove a number of results:

- For Fisher's model with homothetic preferences: if consumers are subject to a *uniform* tax regime, the model loses the *representative consumer* and becomes prone to multiple disconnected equilibria; however we show that the distortion has a structure which leads to two (as opposed to one) representative consumers. We take advantage of this property to develop polynomial time algorithms, in spite of the presence of multiple disconnected equilibria.
- To obtain the result above, we develop a technique to estimate the sensitivity of Fisher's equilibrium prices with respect to the consumers' income, which might find wider applications in the field.
- For the exchange model with Cobb-Douglas utility functions: if consumers are subject to a *differentiated* tax regime, we show that the model is equivalent to a Cobb-Douglas exchange economy with an extra good. This implies that the polynomial time computability is preserved.
- For the exchange model with utility functions satisfying weak gross substitutability: if consumers are subject to a *differentiated* tax regime, the techniques based on convex programming seem to break down. However we are able to modify certain combinatorial algorithms and price adjustment methods to obtain polynomial time approximation schemes in two special cases: an economy with linear utilities and an economy where the demands do not change too rapidly as a function of the prices.

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1 Introduction

The equilibrium problem for a pure exchange economy amounts to finding a set of prices and allocations of goods to economic agents such that each agent maximizes her utility, subject to her budget constraints, and the market clears. The equilibrium depends only on the agents' utility functions and initial endowments of goods.

If one aims at analyzing equilibrium problems arising from real world applications, the scenario outlined above has often to be extended. Indeed one needs to take into account the presence of suitable distortions, which might be, depending on the specific application, transaction costs, transportation costs, tariffs, and/or taxes.

In these frameworks, which are the standard ones for applied computational work, one has to deal with equilibrium conditions influenced by additional parameters which often change the mathematical properties of the problem. For instance, in models with taxes, (i) the equilibrium allocations might lose their Pareto optimality; (ii) restrictions which imply, in the absence of taxes, the uniqueness of equilibrium prices might become compatible with multiple disconnected equilibria.

In this paper, we consider exchange economies with either uniform or differentiated *ad valorem* taxes (see Section 2 for the appropriate definitions). We explore the effects of such tax distortions on models which admit - in the absence of taxes - polynomial time algorithms. In spite of the loss of certain structural properties (including uniqueness), we are able to obtain polynomial time algorithms or approximation schemes in several instances where the model without taxes admitted them.

Background. We now describe the model of an exchange economy, and provide some basic definitions. Let us consider m economic agents which represent traders of n goods. Let $\mathbf{R}^n_+$ denote the subset of $\mathbf{R}^n$ with all nonnegative coordinates. The j-th coordinate in $\mathbf{R}^n$ will stand for good j. Each trader i has a concave utility function $u_i : \mathbf{R}^n_+ \rightarrow \mathbf{R}_+$, which represents her preferences for the different bundles of goods, and an initial endowment of goods $w_i = (w_{i1}, \ldots, w_{in}) \in \mathbf{R}^n_+$. At given prices $\pi \in \mathbf{R}^n_+$, trader i will demand a bundle of goods $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n_+$ which maximizes $u_i(x)$ subject to the budget constraint $\pi \cdot x \leq \pi \cdot w_i$. Let $W_j = \sum_i w_{ij}$ denote the total amount of good j in the market.

An equilibrium is a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ at which, for each trader i, there is a bundle $\bar{x}_i = (\bar{x}_{i1}, \ldots, \bar{x}_{in}) \in \mathbf{R}^n_+$ of goods such that the following two conditions hold: (i) for each trader i, the vector $\bar{x}_i$ maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \leq \pi \cdot w_i$ and $x \in \mathbf{R}^n_+$; (ii) for each good j, $\sum_i \bar{x}_{ij} \leq W_j$.

The celebrated result of Arrow and Debreu [2] states that, under quite mild assumptions, such an equilibrium exists.

We now give the definition of approximate equilibrium. We assume that all the utility functions $u()$ discussed in this paper satisfy $u(0) = 0$.

A bundle $x_i \in \mathbf{R}^n_+$ is an *ε-approximate demand*, for $0 < \varepsilon < 1$, of trader i at prices π if $u_i(x_i) \geq (1-\varepsilon)u^*$ and $\pi \cdot x_i \leq (1+\varepsilon)\pi \cdot w_i$, where $u^* = \max\{u_i(x) | x \in \mathbf{R}^n_+, \pi \cdot x \leq \pi \cdot w_i\}$.

A price vector $\pi \in \mathbf{R}^n_+$ is an ε-approximate equilibrium if there is a bundle x_i for each i such that (1) for each trader i, x_i is an ε-approximate demand of trader i at prices π, and (2) $\sum_i x_{ij} \leq (1+\varepsilon)\sum_i w_{ij}$ for each good j.

An important special case of an exchange economy is the *distributional economy*, where the initial endowments are all *collinear*, i.e., $w_i = \delta_i w$, $\delta_i > 0$, so that the relative incomes of the traders are independent of the prices. This special case is equivalent to *Fisher model*, which is a market of n goods desired by m utility maximizing buyers with fixed incomes.

For any price vector π, the vector $x_i(\pi)$ that maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \leq \pi \cdot w_i$ and $x \in \mathbf{R}^n_+$ is called the *demand* of the i-th trader. By adding up the traders' demands, one gets the *market demand*.

The utility function of an individual trader is said to satisfy *weak gross substitutability* if increasing the prices of some of the goods while keeping some others and her income fixed cannot cause a decrease in demand for the goods whose price is fixed. It is well known and easy to see that the market demand satisfies weak gross substitutability if the utility function of each individual trader does.

A utility function $u(\cdot)$ is *homogeneous* of degree one if it satisfies $u(\alpha x) = \alpha u(x)$, for all $\alpha > 0$, while it is log-homogeneous if it satisfies $u(\alpha x) = u(x) + \log \alpha$, for all $\alpha > 0$.

Both homogeneous and log-homogeneous utility functions represent consumers with *homothetic* preferences, i.e., a bundle x is preferred to a bundle y if and only if the bundle αx is preferred to the bundle αy, for all $\alpha > 0$.

A linear utility function has the form $u_i(x) = \sum_j a_{ij} x_{ij}$. A CES (constant elasticity of substitution) utility function has the form $u(x_i) = (\sum_j (a_{ij} x_{ij})^\rho)^{1/\rho}$, where $-\infty < \rho < 1$, $\rho \neq 0$. The Cobb-Douglas utility function has the form $u_i(x) = \prod_j (x_{ij})^{a_{ij}}$, where $a_{ij} \geq 0$ and $\sum_j a_{ij} = 1$.

Related Work. Substantial work has been done on extending equilibrium models to handle scenarios where good purchases are subject to taxation [19, 20, 25, 26, 27]. Such efforts have provided existential results [25, 26], evidence of tax-induced multiplicity [27] and of the loss of Pareto-optimality of equilibria [19, 20]. Building upon this body of results, applied models have been designed to explicitly take into account tax distortions (see for instance the popular GAMS-MPSGE programming environment).

Previous work within theoretical computer science, which was initiated by [9], has been focussed on restrictions under which the market equilibrium problem, in its version without taxes, can be solved in polynomial time. These restrictions include (i) distributional economies (the Fisher setting) where the traders have homogeneous utility functions [10, 18, 8, 13], (ii) exchange economies which satisfy *weak gross substitutability* [16, 15, 7, 4], and (iii) exchange economies with some families of CES and nested CES utility functions [3, 17]. For a more comprehensive list of results, see [6].

Our Results. In this paper we analyze the effects of tax distortions on scenarios which, in the absence of taxes, admit polynomial time algorithms. First of all, an exchange economy with uniform ad valorem taxes can be efficiently transformed into an equivalent exchange economy without taxes, with the same utility functions and different initial endowments, obtained by redistributing the original ones (Section 2.1). Therefore algorithms for the exchange model that do not depend on the distribution of endowments carry through the model with uniform taxes.

We prove that a distributional economy (which is equivalent to Fisher's model) with uniform ad valorem taxes and homothetic consumers can be efficiently transformed into an equivalent two-trader exchange economy without taxes (Section 3.1). We then develop a polynomial-time algorithm for approximating the equilibrium (Section 3.2). To analyze some parameters related to the accuracy of the algorithm, we use the tool of implicit differentiation, which we believe will find more general applications (Section 3.4). Note that a two-trader exchange economy with homothetic consumers admits multiple disconnected equilibria [14]. The example in [14] can be modified to model an exchange economy that is equivalent to a distributional economy with uniform ad valorem taxes. Therefore our algorithm provides the first significant example of polynomial time computation of equilibria in a setting with multiple disconnected equilibria.

We then show that an n-good exchange economy with differentiated ad valorem taxes and m Cobb-Douglas consumers can be efficiently transformed into an equivalent $(n+1)$-good exchange economy without taxes, with m Cobb-Douglas consumers (Section 2.2). Since the equilibrium for a Cobb-Douglas exchange economy can be computed in polynomial time [11], our reduction shows that the same is possible for the model with non-uniform taxation.

We believe that this reduction is not possible for more general utility functions, including some of the commonly studied ones. Consequently, we do not at present have polynomial time algorithms or approximation schemes for the model with non-uniform taxes when the traders have CES functions with $-1 \leq \rho < 0$, even though there are explicit convex programs and polynomial time algorithms for the corresponding pure exchange model [3]. The same happens for certain nested CES functions [17]. The situation is notably better when the utility functions satisfy *weak gross substitutability*. Here too, methods based on explicit or implicit convex programs seem to be unavailable, in contrast with the pure exchange model. This is true even for the case of linear utility functions, where explicit convex programs are known for the pure exchange model [23, 15]. Interestingly, in spite of this, we are able to show that the combinatorial algorithm of Garg and Kapoor [13] for linear utilities can be adapted to handle the differentiated tax scenario (Section 4). We also consider the case of economies with *weak gross substitutability* and demand functions that do not change too rapidly with prices. In this case, it is possible to obtain a poly-time approximation scheme via a simple price adjustment (tâtonnement) scheme. The main tool is Lemma 7 from [1], which guarantees convergence of a *normalized version* of tâtonnement, where the price of one good (called the *numéraire*) is kept fixed. By expressing the role of taxation in terms of a fictitious good (the "tax good" of Section 2.2) which acts as the numéraire, we can then analyze the convergence of the normalized process. The lemma above becomes thus relevant because it does not require gross substitutability with respect to the good whose price is not part of the dynamics. For lack of space, this last result is omitted from the extended abstract.

2 Exchange Economies with Taxes on Consumption

We describe the model of an exchange economy with *ad valorem* taxes as presented by Kehoe ([19], pp. 2127-2128), and distinguish between the uniform case, where the tax rate is uniform across consumers, and the non-uniform case, where the tax rate is differentiated among consumers.

2.1 Uniform Ad Valorem Taxes

Consider a trader i with utility function $u_i(x_i)$ and initial endowment w_i.

Let $\tau_j \geq 0$ be the uniform *ad valorem* tax associated with the consumption of good j. This means that if consumer i purchases x_{ij} units of good j at price π_j, she will spend on this good the amount $\pi_j x_{ij} + \tau_j \pi_j x_{ij}$.

We postulate the presence of a special actor, the *government*, which will rebate the tax revenues to consumers. Let $\theta_i \geq 0$, with $\sum_i \theta_i = 1$, be the share of total tax revenues rebated to consumer i as a lump sum.

Then the classical consumer's maximization problem gets modified as follows:

$$\max u_i(x_i) \tag{1}$$

$$\text{s.t. } \sum_j \pi_j (1+\tau_j) x_{ij} \le \sum_j \pi_j w_{ij} + \theta_i R, \tag{2}$$

where $x_i \in \mathbf{R}^n_+$, and R is the total amount of revenues distributed by the government.

In this context, the market equilibrium problem consists of finding $(\bar{\pi}, \bar{x_i}, \bar{R})$ such that

- at prices $\bar{\pi}$, $\bar{x_i}$ solves (1) (2), $\forall i$ (optimality and budget constraint are satisfied for all consumers);
- $\sum_i \bar{x}_{ij} = \sum_i w_{ij}$, $\forall j$ (the market clears all the goods);
- $\bar{R} = \sum_j \bar{\pi}_j \tau_j \sum_i \bar{x}_{ij}$ (the amount of taxes distributed is equal to the amount of taxes collected).

We now show that the equilibria for such an economy are in a one-to-one correspondence with the equilibria of an exchange economy without taxes, where the traders have a different set of initial endowments, obtained by a suitable redistribution of the original ones. This correspondence has been established in [5], where models of taxation were one of the targets of some experimental work.

Whenever the equilibria of two economies are in a one-to-one correspondence, and can be immediately computed one from the other, we say that the two economies are equivalent.

Proposition 1 *[5] Let $E = E(u_i(\cdot), w_i, \tau, \theta)$ be an exchange economy with uniform ad valorem taxes $\tau = (\tau_1, \ldots, \tau_n)$, and tax shares $\theta = (\theta_1, \ldots, \theta_m)$. E is equivalent to an exchange economy without taxes $E' = E'(u_i(\cdot), w'_i)$ where $w'_{ij} = \frac{w_{ij}}{1+\tau_j} + \theta_i \frac{\tau_j}{1+\tau_j} \sum_i w_{ij}$.*

Proof : The proof is in Appendix A. □

2.2 Differentiated Taxes

We now consider a more general model in which the taxes on purchases are different for each trader. We call this scheme *specific taxation* or *differentiated ad valorem taxation.*

In an exchange economy with specific taxes, $\tau_{ij} \ge 0$ is the tax rate imputed to trader i on purchase of good j. The setting for an exchange economy with specific taxes differs from that with uniform taxes in the budget constraint, which is now given by

$$\sum_j \pi_j (1+\tau_{ij}) x_{ij} \le \pi \cdot w_i + \theta_i R\,. \tag{3}$$

At equilibrium, we must have $R = \sum_{ij} \pi_j \tau_{ij} x_{ij}$.

The lack of uniformity of this model, which differentiates between consumers, prevents the possibility of a direct reduction to a pure exchange economy, obtained by redistributing the individual endowments, as in Proposition 1. Nevertheless, this model can be made *similar* to a pure exchange economy with an extra good. Indeed, note that the budget constraint can be rewritten as

$$\pi \cdot x_i + R x_{i,n+1} \le \pi \cdot w_i + R\theta_i \tag{4}$$

where $x_{i,n+1} = \frac{\sum_j \pi_j \tau_{ij} x_{ij}}{R}$.

If we interpret R as the price of an additional (fictitious) good, that we call the "tax good", and θ_i and $x_{i,n+1}$ as the i-th trader's initial endowment and demand of such good, then inequality (4) corresponds to the budget constraint of a consumer in a pure exchange economy with an extra good.

To get a reduction to an exchange economy without taxes, one would now need to exhibit a utility function, defined on $n+1$ goods, which, combined with the budget constraint (4), gives a demand of $x_{ij}(\pi, R)$ for the first n goods, and $x_{i,n+1}(\pi, R) = \frac{\sum_j \pi_j \tau_{ij} x_{ij}}{R}$ for the "tax good".

We do not know if such a reduction is possible in general. However we show below (Proposition 2) that it can be done in the case of exchange economies where the traders have Cobb-Douglas utility functions.

Proposition 2 *Let n and m be the number of goods and the number of traders, respectively. Let $u_i(\cdot)$, $i = 1, \ldots, m$, be Cobb-Douglas utility functions. We denote by $E_{n,m} = E_{n,m}(u_i(\cdot), w_i, \tau_i, \theta)$ a Cobb-Douglas exchange economy with differentiated ad valorem taxes $\tau_i = (\tau_{i1}, \ldots, \tau_{in})$, and tax shares $\theta = (\theta_1, \ldots, \theta_m)$. $E_{n,m}$ is equivalent to an exchange economy without taxes $E'_{n+1,m} = E'_{n+1,m}(v_i(\cdot), w'_i)$ where the $v_i(\cdot)$'s are Cobb-Douglas utility functions, and $w'_i = (w_{i1}, \ldots, w_{in}, \theta_i)$.*

Proof : The proof is in Appendix B. □

3 Collinear endowments distorted by uniform taxation

The general reduction of Proposition 1 shows that uniform taxation does not affect algorithms which compute equilibrium prices for exchange economies without exploiting any particular property of the initial endowments. Therefore, several results for pure exchange economies extend to the model with uniform taxation. One interesting case where the redistribution of initial endowments potentially carries negative computational consequences is that of exchange economies with collinear endowments and homothetic preferences. In this setting an equilibrium (without taxes) can be computed in polynomial time by convex programming [8], based on certain aggregation properties of the economy which imply the existence of a *representative consumer* [12]. The redistribution of endowments associated with taxation clearly destroys the collinearity of endowments, and thus the collapse to a single consumer's problem. We show that in an exchange economy with collinear endowments and homothetic consumers, the model with uniform taxation is equivalent to an exchange economy with *two representative consumers*. Building upon this property, we then show how to compute an approximate equilibrium in polynomial time for a wide family of problems.

3.1 Reduction to two representative consumers

Recall that a *distributional economy* is an exchange economy where the initial endowments of the traders are collinear. In other words, the k-th trader has endowments of the form $w_k = \gamma_k w$, for $k = 1, \ldots, m$, where $w = (w_1, \ldots, w_n)$ describes the overall amount of each good in the market, and γ_k is a positive constant less then one. We have $\sum_k \gamma_k = 1$. In this scenario, the relative incomes of the traders are constants, so that the model is equivalent to Fisher's model.

If we specialize the model of Section 2.1 to an economy with collinear initial endowments, then the redistribution described by Proposition 1, gives $w'_{ij} = \gamma_i \frac{w_j}{1+\tau_j} + \theta_i \frac{\tau_j}{1+\tau_j} w_j$.

Notice that the matrix whose columns represent the new initial endowments of the traders has rank at most two. Indeed all the columns are linear combinations of the vectors z and s, whose j-th entries are $z_j = \frac{w_j}{1+\tau_j}$, and $s_j = \frac{\tau_j}{1+\tau_j} w_j$, respectively. Note that $w'_i = \gamma_i z + \theta_i s$. (Note also that $w = z + s$ and verify that $\sum_i w'_i = w$.) Thus the effect of uniform taxation on economies with proportional endowments amounts to an increase of the rank of the endowment matrix from one to two.

Whenever the consumers are homothetic, exchange economies with a rank two endowment matrix can be reduced to a two-trader economy, according to the following scheme:

1. Let z be the n-vector whose j-th component is $\frac{w_j}{1+\tau_j}$, and s be the n-vector whose j-th component is $\frac{w_j \tau_j}{1+\tau_j}$.
2. For all k, split the k-th trader into two traders, which have the same utility function of the original trader and initial endowments $\gamma_k z$, and $\theta_k s$, respectively. This procedure produces two groups of m traders each, where the traders in each group have proportional endowments.
3. Based on the properties in [12], aggregate all the consumers from each group into one representative consumer, with endowment given by the sum of their endowments and utility function obtained by aggregating the utility functions as in [12]. This gives a two-trader economy.

These arguments lead to the following result.

Proposition 3 *Let $u_i(\cdot)$, $i = 1, \ldots, m$, be log-homogeneous utility functions. Let $E_m = E(u_i(\cdot), w, \tau, \theta, \gamma)$ be an m-trader distributional economy with uniform ad valorem taxes $\tau = (\tau_1, \ldots, \tau_n)$, tax shares $\theta = (\theta_1, \ldots, \theta_m)$, and income shares $\gamma = (\gamma_1, \ldots, \gamma_m)$. E_m is equivalent to a two-traders exchange economy without taxes $E'_2 = E'_2(v_1(\cdot), v_2(\cdot), z, s)$, where v_1 and v_2 are the log-homogeneous utility functions of the two consumers, and z and s are their initial endowment vectors. Here, $v_1(x)$ (resp. $v_2(x)$) is defined to be the maximum of $\sum_i \gamma_i u_i(x_i)$ (resp. $\sum_i \theta_i u_i(x_i)$) over all $x_1, \ldots, x_m \in \mathbf{R}^n_+$ such that $\sum_i x_i = x$.*

3.2 The algorithm

The reduction summarized in Proposition 3, combined with some results of Mantel [21] on two-trader economies, suggest the following algorithm, which we call RSR (Reduce-Solve-Reconstruct), for the computation of an approximate equilibrium. For the analysis, we assume that $\tau_j > 0$ for each j. Let $\tau_{min} = \min_j \tau_j$ and $\tau_{max} = \max_j \tau_j$.

Algorithm RSR

1. The input is given in terms of m log-homogeneous utility functions u_i, $i = 1, \ldots, m$, and vectors w, γ, θ, τ.
2. Apply the transformation of Proposition 3, which returns an economy with two homothetic consumers (with utility functions v_1 and v_2, and initial endowments z and s) and n goods.
3. Consider the following constrained maximization problem:

$$\begin{aligned} \max \quad & \alpha v_1(x_1) + (1-\alpha) v_2(x_2) \\ \text{s.t.} \quad & x_1 + x_2 = w \\ & x_1,\, x_2 \geq 0 \end{aligned}$$

For a given $0 \le \alpha \le 1$, let $x_1(\alpha)$ and $x_2(\alpha)$ be maximizing allocations, and let $\pi(\alpha)$ be the vector of shadow prices (Lagrange multipliers). It can be shown that $\pi(\alpha) \cdot x_1(\alpha) = \alpha$, $\pi(\alpha) \cdot x_2(\alpha) = 1 - \alpha$, and thus $\pi(\alpha) \cdot w = 1$. Moreover, $x_1(\alpha)$ and $x_2(\alpha)$ have the "right shape" – they are proportional to the optimal bundles demanded by the two traders at the price $\pi(\alpha)$. Let $B_1(\alpha) = \pi(\alpha) \cdot (z - \bar{x}_1(\alpha))$ and $B_2(\alpha) = \pi(\alpha) \cdot (s - \bar{x}_2(\alpha))$ be the functions expressing the (positive or negative) *savings* of consumer 1 and 2, respectively. Note that $B_1(\alpha) + B_2(\alpha) = 0$, since $z + s = w$. Thus, if $B_1(\alpha) = 0$, then we have $\pi(\alpha) \cdot x_1(\alpha) = \pi(\alpha) \cdot z$, and $\pi(\alpha) \cdot x_2(\alpha) = \pi(\alpha) \cdot s$. Thus, $x_1(\alpha)$ and $x_2(\alpha)$ are not merely proportional to the optimal bundles, but they *are* the optimal bundles of the two traders at price $\pi(\alpha)$. Thus $\pi(\alpha)$ is an equilibrium for the two trader economy [22].

We therefore find an approximate equilibrium for the two-trader economy by finding a value of α such that $B_1(\alpha)$ and $B_2(\alpha)$ are sufficiently close to zero. In such a case, $\pi(\alpha)$, $x_1(\alpha)$ and $x_2(\alpha)$ form an approximate equilibrium, provided that the functions $B_i(\alpha)$ are smooth enough (see the extensive discussion below). The search for an appropriate value of α can be done by the bisection method, i.e., binary search, guided by the value of $B_i(\alpha)$, computed from the values $x_i(\alpha)$ and $\pi(\alpha)$ returned by the solution of the maximization problem above. The applicability of bisection method builds upon some results by Mantel [21] on the global convergence of the welfare adjustment process when applied to two-trader economies.

4. From the solution to the two-trader problem, reconstruct the solution to the $2m$-trader problem, i.e., the corresponding allocations, and then to the m-trader problem without taxes.

5. Compute approximate equilibrium prices $\tilde{\pi}_j$, $j = 1, \ldots$, for the original economy with taxes, by scaling prices $\pi_j(\alpha)$, i.e., $\tilde{\pi}_j = \frac{\pi_j(\alpha)}{1+\tau_j}$.

3.3 Analysis of algorithm RSR

For any price vector π, it is easy to see that $\frac{\pi \cdot z}{\pi \cdot w}$ lies in the interval $[\alpha_{min} = \frac{1}{1+\tau_{max}}, \alpha_{max} = \frac{1}{1+\tau_{min}}]$. Thus if $B_1(\bar{\alpha}) = 0$, then $\bar{\alpha} = \frac{\pi(\bar{\alpha}) \cdot x_1(\bar{\alpha})}{\pi(\bar{\alpha}) \cdot w} = \frac{\pi(\bar{\alpha}) \cdot z}{\pi(\bar{\alpha}) \cdot w}$ lies in the range $[\alpha_{min}, \alpha_{max}]$. It is also easy to verify that $B_1(\alpha_{min}) \ge 0$ and $B_1(\alpha_{max}) \le 0$. So we perform our binary search in the interval $[\alpha_{min}, \alpha_{max}]$.

The binary search is described and analyzed in Appendix C, where we show that Algorithm RSR computes an ε-approximate equilibrium in time of the order of $T(n)(\log M + \log \frac{1}{\varepsilon} + \log \frac{1}{\alpha_{min}} + \log \frac{1}{(1-\alpha_{max})})$, where $T(n)$ is the polynomial bound on the time required to solve the convex program, and M is an upper bound on the absolute value of the derivative of $B_1(\alpha)$ in the interval $[\alpha_{min}, \alpha_{max}]$. The next section takes a close look at the parameter M that influences the running time.

3.4 Sensitivity of the welfare maximization problem

The running time of algorithm RSR depends on the logarithm of M, where M is an upper bound on $|B_1'(\alpha)|$ for α in $[\alpha_{min}, \alpha_{max}]$. We now show how to estimate M for a wide family of utility functions.

The two-trader maximization problem occurring in step 3 of Algorithm RSR, when written in its expanded form, becomes the $2m$-trader maximization problem:

$$\begin{aligned} \max \quad & \alpha \sum_i \gamma_i u_{1i}(x_i^1) + (1-\alpha) \sum_i \theta_i u_{2i}(x_i^2) \\ \text{s.t.} \quad & x_1^1 + \ldots + x_m^1 + x_1^2 + \ldots + x_m^2 = w \\ & x_i^1, \, x_i^2 \ge 0 \end{aligned}$$

where $u_{1i}() = u_{2i}() = u_i()$. Let $x_i^\ell(\alpha)$ denote the solution of this problem, and $\pi(\alpha)$ the corresponding shadow prices. We will simply denote $x_i^\ell(\alpha)$ by x_i^ℓ and $\pi(\alpha)$ by p. Since $B_1(\alpha) = p \cdot z - \alpha$, we can upper bound $|B_1'(\alpha)|$ by bounding the elements of the vector $\frac{\partial p}{\partial \alpha}$.

Let us consider the first order optimality conditions for the problem above, which we will denote by

$$H(\overbrace{x, p}^{\text{y}}, \alpha) = 0.$$

These equations can be explicitly written as:

$$\begin{cases} \alpha\gamma_i \nabla u_{1i}(x_i^1) - p = 0\,; \\ (1-\alpha)\theta_i \nabla u_{2i}(x_i^2) - p = 0\,; \\ x_1^1 + \ldots + x_m^1 + x_1^2 + \ldots + x_m^2 - w = 0. \end{cases}$$

By *Implicit Differentiation*, from the first order conditions we obtain the following equation:

$$\nabla_y H(x,p,\alpha)\frac{\partial}{\partial \alpha} y(\alpha) + \frac{\partial}{\partial \alpha} H(x,p,\alpha) = 0. \qquad (5)$$

Let A_i and B_i denote the Hessian $\alpha\gamma_i\nabla^2 u_{1i}(x_i^1)$, and $(1-\alpha)\theta_i\nabla^2 u_{2i}(x_i^2)$, respectively. Equation 5 takes the form:

$$\begin{bmatrix} A_1 & & & & & & -I \\ & \ddots & & & & & \vdots \\ & & A_m & & & & -I \\ & & & B_1 & & & -I \\ & & & & \ddots & & \vdots \\ & & & & & B_m & -I \\ I & \ldots & I & I & \ldots & I & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x_1^1}{\partial \alpha} \\ \vdots \\ \frac{\partial x_m^1}{\partial \alpha} \\ \frac{\partial x_1^2}{\partial \alpha} \\ \vdots \\ \frac{\partial x_m^2}{\partial \alpha} \\ \frac{\partial p}{\partial \alpha} \end{bmatrix} + \begin{bmatrix} \gamma_1 \nabla u_{11}(x_1^1) \\ \vdots \\ \gamma_m \nabla u_{1m}(x_m^1) \\ -\theta_1 \nabla u_{21}(x_1^2) \\ \vdots \\ -\theta_m \nabla u_{2m}(x_m^2) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Assume that the A_i's and B_i's be nonsingular, i.e., that they are negative definite. In this case the matrix of the linear system above has an inverse, which is

$$\begin{bmatrix} A_1^{-1} - A_1^{-1}KA_1^{-1} & -A_1^{-1}KA_2^{-1} & \cdots & -A_1^{-1}KB_m^{-1} & A_1^{-1}K \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ -B_m^{-1}KA_1^{-1} & \ldots & \ldots & B_m^{-1} - B_m^{-1}KB_m^{-1} & B_m^{-1}K \\ -KA_1^{-1} & \ldots & \ldots & -KB_m^{-1} & K \end{bmatrix},$$

where $K = \left(A_1^{-1} + \ldots + A_m^{-1} + B_1^{-1} + \ldots + B_m^{-1}\right)^{-1}$.

Let now $d_{1i} = \gamma_i \nabla u_{1i}(x_i^1)$, and $d_{2i} = \theta_i \nabla u_{2i}(x_i^2)$. We obtain the expression

$$\frac{\partial p}{\partial \alpha} = -K(\sum_i A_i^{-1} d_{1i} - \sum_i B_i^{-1} d_{2i}), \qquad (6)$$

from which we can upper bound the absolute value of any element of $\frac{\partial p}{\partial \alpha}$ in terms of $\|A_i^{-1}\|$, $\|B_i^{-1}\|$, $\|K\|$, and $\nabla u_{\ell i}(x_i^\ell)$, where $\|\cdot\|$ denotes the spectral norm of a matrix, which, in the case of semi-definite matrices, coincides with the spectral radius.

In order to bound the spectral norm of K in terms of those of A_i and B_i, we now use a classical result from linear algebra. The theorem we need (see [24], p.192) states that if C_1 and

C_2 are real symmetric matrices with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, respectively, then the eigenvalues $s_1 \geq s_2 \geq \ldots \geq s_n$ of $C_1 + C_2$ satisfy $\lambda_k + \mu_n \leq s_k \leq \lambda_k + \mu_1$.

For any matrix Q, let $\lambda_n(Q)$ denote its smallest eigenvalue. Then, if Q is positive definite, the spectral norm of its inverse is given by $\frac{1}{\lambda_n(Q)}$. We apply the theorem to the positive definite matrix $-K$, and obtain (the calculation is shown for $m = 1$, for the sake of clarity)

$$\left\|(-A_1^{-1} - B_1^{-1})^{-1}\right\| = \lambda_{max}(-A_1^{-1} - B_1^{-1})^{-1} = \frac{1}{\lambda_{min}(-A_1^{-1} - B_1^{-1})} \leq$$

$$\leq \frac{1}{\lambda_{min}(-A_1^{-1}) + \lambda_{min}(-B_1^{-1})} = \frac{1}{\frac{1}{\|A_1\|} + \frac{1}{\|B_1\|}} \leq \min\{\|A_1\|, \|B_1\|\}.$$

Let now $\|A_m\| = \min_i \|A_i\|$, $\|B_m\| = \min_i \|B_i\|$, $\left\|A_M^{-1}\right\| = \max_i \left\|A_i^{-1}\right\|$, and $\left\|B_M^{-1}\right\| = \max_i \left\|B_i^{-1}\right\|$. From Equation 6, we then get the crude upper bound

$$\left|\frac{\partial p_j}{\partial \alpha}\right| \leq 2md \min\{\|A_m\|, \|B_m\|\} \max\{\left\|A_M^{-1}\right\|, \left\|B_M^{-1}\right\|\},\ j = 1, \ldots, m,$$

where d is the maximum of the $d_{\ell i}$'s.

We have shown how to bound $|B_1'(\alpha)|$ in terms of the Hessians of the utility functions $u_{\ell i}$ evaluated at x_i^ℓ. For utility functions with negative definite Hessians, this gives an approach for bounding $|B_1'(\alpha)|$. We show how we can proceed even further for an interesting class of such functions. (Incidentally, recall that negative definiteness of the Hessian is a sufficient condition for strict concavity of the utility function, but it is not necessary.)

Let $f(x_1, \ldots, x_n)$ be a concave log-homogeneous utility function. Its Hessian $F(x_1, \ldots, x_n)$ is negative semi-definite. Consider the function $g(x_1, \ldots, x_n) = \frac{1}{n}(\log x_1 + \ldots + \log x_n)$. Its Hessian $G(x_1, \ldots, x_n)$ is the diagonal matrix whose (i,i)-th entry is $-\frac{1}{nx_i^2}$, so that the maximum eigenvalue of $-G$ is at most $\frac{1}{nc^2}$, where c is the minimum consumption of any good in x, and its minimum eigenvalue is at least $\frac{1}{nC^2}$, where C is the maximum consumption.

Let now consider the function $h(x_1, \ldots, x_n) = (1-\sigma)f(x_1, \ldots, x_n) + \sigma g(x_1, \ldots, x_n)$, for a small positive constant σ less than one. This function is log-homogeneous, and its Hessian $H(x_1, \ldots, x_n)$ is negative definite (again by applying the result in [24]).

Let $f_{max} = \max_{ij}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right|$. We obtain:

$$\|H\| \leq (1-\sigma)nf_{max} + \frac{\sigma}{nc^2},\ \ \left\|H^{-1}\right\| \leq \frac{1}{(1-\sigma)\lambda_{min}(-F) + \sigma\lambda_{min}(-G)} \leq \frac{nC^2}{\sigma}. \tag{7}$$

Therefore the function h (which is intended as an approximation of the concave log-homogeneous function f) is such that: (i) the Hessian H is a negative definite matrix (if there is positive consumption of each of the goods);(ii) the spectral norm of H can be bounded in terms of the minimum consumption, and the absolute value of the maximum entry of F, while the spectral norm of its inverse can be bounded in terms of the maximum consumption.

In order to lower bound the minimum consumption c we can proceed as follows. Let us consider the maximization problem of a consumer with a utility function of the form $h(x) = (1-\sigma)f(x) + \sigma g(x)$, where f and g are as above, and budget constraint $\pi \cdot x = I$. The first order conditions for this problem are of the form

$$(1-\sigma)\frac{\partial f(x)}{\partial x_k} + \sigma\frac{1}{nx_k} = \lambda\pi_k.$$

If we multiply both sides of this equation by x_k, and then sum over k, we obtain that $\lambda = \frac{1}{I}$. Since $\frac{\partial f(x)}{\partial x_k} \geq 0$, we get $x_k \geq I \frac{\sigma}{n\pi_k}$. Since $\alpha \in [\alpha_{min}, \alpha_{max}]$, the minimum expenditure I_{min} over all traders at α is bounded below by $\min_i \min\{\gamma_i \alpha_{min}, \theta_i(1-\alpha_{max})\}$. Since the total expenditure of the $2m$ traders in the welfare maximization problem above is one, then $p_k w_k \leq 1$, from which we must have $x_k \geq I \frac{\sigma w_k}{n}$. Therefore, the minimum consumption is lower bounded by $\frac{\sigma I_{min} w_{min}}{n}$, where w_{min} is the quantity of the most scarce good.

The maximum consumption C can be upper bounded by observing that $\pi_j x_j \leq 1$. Now notice that at least one of the $2m$ traders has an expenditure of at least $1/2m$. This trader will demand at least $\frac{\sigma}{2\pi_j nm}$ units of good j. If π is to be an equilibrium, we must have $\frac{\sigma}{2\pi_j nm} \leq w_j$. Therefore we obtain $C \leq \frac{2nmw_{max}}{\sigma}$, where w_{max} the quantity of the most abundant good.

Finally, for a wide family of utility functions, we can upper bound f_{max} in terms of minimum and maximum consumptions. The reader is invited to check this when f is the log of a CES function.

4 Linear Exchange Economies with Differentiated Taxes

We consider an economy with m traders and n goods where each trader has a linear utility function. Let $u_i = \sum_j a_{ij} x_j$ denote the utility function of the i-th trader. Each trader has the initial endowment $w_i \in \mathbf{R}^n_+$. Let $\tau_{ij} \geq 0$ denote the tax rate of the i-th trader for the consumption of the j-th good. And let θ_i denote the share of the i-th trader in the overall tax collected. We have $\sum_i \theta_i = 1$. An equilibrium is a price vector $\pi = (\pi_1, \ldots, \pi_n)$ and a number $R \geq 0$ at which there are bundles $x_i \in \mathbf{R}^n_+$ for each trader i such that (1) x_i maximizes $u_i(x)$ over all $x \in \mathbf{R}^n_+$ such that the cost $\sum_j \pi_j(1+\tau_{ij})x_j$ of bundle x is at most the income $\theta_i R + \sum_j \pi_j w_{ij}$; (2) $\sum_i x_{ij} \leq \sum_i w_{ij}$ for each good j; and (3) $\sum_i T_i(x_i, \pi) = R$, where $T_i(x, \pi)$ is defined to be $\sum_j \tau_{ij}\pi_j x_j$, the tax that i has to pay to consume x at price π.

For $\varepsilon > 0$, we define an ε-approximate equilibrium to be a price vector $\pi = (\pi_1, \ldots, \pi_n)$ and a number $R \geq 0$ at which there are bundles $x_i \in \mathbf{R}^n_+$ for each trader i such that (1) $\pi \cdot x_i + T(x_i, \pi) \leq (1+\varepsilon)(\theta_i R + \sum_j \pi_j w_{ij})$, and $u_i(x_i) \geq (1-\varepsilon) v_i(\pi, R)$, where $v_i(\pi, R)$ is the maximum value of $u_i(x)$ over all $x \in \mathbf{R}^n_+$ such that $\sum_j \pi_j(1+\tau_{ij})x_j \leq \theta_i R + \sum_j \pi_j w_{ij}$; (2) $\sum_i x_{ij} \leq (1+\varepsilon)\sum_i w_{ij}$ for each good j; and (3) $(1-\varepsilon)R \leq \sum_i T_i(x_i, \pi) \leq (1+\varepsilon)R$.

We now describe our algorithm, an adaptation of the auction based algorithm of Garg and Kapoor [13], for computing an approximate equilibrium of the model. The analysis of the algorithm assumes that $a_{ij} > 0$ for each i and j. Let τ_{max} denote $\max_{i,j} \tau_{ij}$. For simplifying some expressions, we also assume, without loss of generality, that $\sum_i w_{ij} = 1$ for each j. Let $w_{min} = \min_{i,j} w_{ij}$. Our analysis also assumes that $w_{min} > 0$. Let $\kappa = 1/w_{min}$.

Let $\delta = \frac{\varepsilon}{90n(1+\tau_{max})\kappa}$. The algorithm has variables π_j for the prices, and a variable R that stands for the tax money that is distributed to the buyers. The algorithm starts with all prices set to 1. From time to time, it increases the price of some good by a multiplicative factor of $1+\delta$. It has variables y_{ij} and h_{ij} corresponding to the amounts of good j allocated to i at the current price π_j and the previous price $\pi_j/(1+\delta)$, respectively. Let $x_{ij} = y_{ij} + h_{ij}$. Let $T'_i(y_i, h_i, \pi) = \sum_j \tau_{ij}(\pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij})$, the tax that i pays in consuming y_i and h_i. Let

$$D_i(\pi) = \{j | \frac{a_{ij}}{(1+\tau_{ij})\pi_j} \geq \frac{a_{ik}}{(1+\tau_{ik})\pi_k} \text{ for } 1 \leq k \leq n\}.$$

Initialize. Let $\pi_j = 1$ for $1 \leq j \leq n$, $R = 1$, $y_{ij} = 0$ and $h_{ij} = 0$ for each i and j.
Phase 1:

1. We make a call to the procedure Allocatemore(), described below.
2. If $\sum_i T'_i(y_i, h_i, \pi) = R$, let $R \leftarrow R(1+\delta)$ and go to Step 1 of Phase 1.
3. If for each i, we have $\sum_j(\pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij}) + T'_i(y_i, h_i, \pi) = \theta_i R + \sum_j \pi_j w_{ij}$, the algorithm ends.
4. If for some i, we have $\sum_j(\pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij}) + T'_i(y_i, h_i, \pi) < \theta_i R + \sum_j \pi_j w_{ij}$, consider any $j \in D_i(\pi)$. An inspection of Allocatemore() tells us that we must have $h_{i'j} = 0$ for every trader i' and $\sum_{i'} y_{i'j} = 1$. We call Raiseprice(j). If $\pi_k > 0$ for every good k, we jump to Step 1 of Phase 2. Otherwise, return to Step 1 of Phase 1.

Phase 2:

1. If $R \leq \delta \sum_j \pi_j w_{ij}$ for each i, let $R \leftarrow \sum_i T'(y_i, h_i, \pi)$; the algorithm ends.
2. Make a call to Allocatemore().
3. If $\sum_i T'_i(y_i, h_i, \pi) = R$, the algorithm ends.
4. If for each i, we have $\sum_j(\pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij}) + T'_i(y_i, h_i, \pi) = \theta_i R + \sum_j \pi_j w_{ij}$, the algorithm ends.
5. If for some i, we have $\sum_j(\pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij}) + T'_i(y_i, h_i, \pi) < \theta_i R + \sum_j \pi_j w_{ij}$, consider any $j \in D_i(\pi)$. We must have $h_{i'j} = 0$ for every trader i' and $\sum_{i'} y_{i'j} = 1$. We call Raiseprice(j) and return to Step 1 of Phase 2.

To complete the description of the algorithm, we need to specify the two procedures Allocatemore and Raiseprice.

The procedure Allocatemore. In this procedure, we first solve a linear program, which uses as data the current values of the variables π, y, h, and R. The linear program has variables y'_{ij} and h'_{ij} for each trader i and good j. The linear program is:

$$\begin{aligned}
\text{Maximize} \quad & \sum_{i,j} y'_{ij} \\
\text{Subject to} \quad & \\
\sum_i (y'_{ij} + h'_{ij}) &\leq 1 \text{ for each } j \\
\sum_i \sum_j (y'_{ij} + h'_{ij}) &= 1 \text{ for each } j \text{ such that } \pi_j > 1 \\
\sum_i T'_i(y'_i, h'_i, \pi) &\leq R \\
\sum_j \pi_j y'_{ij} + \frac{\pi_j}{1+\delta} h'_{ij} + T'_i(y'_i, h'_i, \pi) &\leq \theta_i R + \sum_j \pi_j w_{ij} \text{ for each } i \\
y'_{ij} &= 0 \text{ for each } i \text{ and } j \notin D_i(\pi). \\
h'_{ij} &= 0 \text{ for each } i \text{ and } j \text{ such that } h_{ij} = 0. \\
y'_{ij} &\geq 0 \text{ for each } i, j \\
h'_{ij} &\geq 0 \text{ for each } i, j
\end{aligned}$$

As we argue below, this linear program will always be feasible. Thus the maximization is well defined. After solving the linear program, we set $y_{ij} = y'_{ij}$ and $h_{ij} = h'_{ij}$ for each i and j. This completes the description of the procedure Allocatemore.

The procedure Raiseprice(j). In this procedure, we set $\pi_j \leftarrow \pi_j(1+\delta)$, $h_{ij} \leftarrow y_{ij}$ for each i, and $y_{ij} \leftarrow 0$ for each i. This completes the description of Raiseprice.

Phases 1 and 2 of our algorithm are similar to the basic algorithm of Garg and Kapoor [13] - a call to Allocatemore() replaces the sequence of steps in their algorithm occurring between two price raises. Our algorithm needs to track the relation of $\sum_i T'(y_i, h_i, \pi)$ – the tax paid as a consequence of consumption – to R, the tax that is distributed as income. The reason we have Phase 2 is that at the end of Phase 1, $\sum_i T'(y_i, h_i, \pi)$ can be significantly smaller than R.

Theorem 4 *For any $\varepsilon > 0$, our algorithm computes an ε-approximate equilibrium for the model in time that is polynomial in the input size, $1/\varepsilon$, and κ.*

For lack of space, the proof of this theorem is given in Appendix D. In the process, we state some invariants that help in understanding the algorithm.

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Appendix A: Proof of Proposition 1

¿From the market clearance condition, it follows that the tax revenue at equilibrium must be

$$R = \sum_j \pi_j \tau_j \sum_i x_{ij} = \sum_j \pi_j \tau_j \sum_i w_{ij}.$$

We can therefore eliminate the equilibrium requirement $R = \sum_j \pi_j \tau_j \sum_i x_{ij}$ by substituting $\sum_j \pi_j \tau_j \sum_i w_{ij}$ for R in the right hand side of inequality (2), which thus becomes $\sum_j \pi_j (w_{ij} + \theta_i \tau_j \sum_i w_{ij})$.

After dividing and multiplying each term of this expression by $1 + \tau_j$, the budget constraint can be rewritten as

$$\sum_j \pi_j (1 + \tau_j) x_{ij} \leq \sum_j \pi_j (1 + \tau_j) w'_{ij}, \text{ where } w'_{ij} = \frac{w_{ij}}{1 + \tau_j} + \theta_i \frac{\tau_j}{1 + \tau_j} \sum_i w_{ij}.$$

We have therefore reduced the equilibrium in the original economy to an equilibrium in a pure exchange economy, where the i-th trader now has an initial endowment w'_i. It is easy to check that $\sum_i w'_{ij} = \sum_i w_{ij}$ for each j, so that the two economies have the same quantity of each good. Finally we have that $(\pi_1, \ldots, \pi_n)$ is an equilibrium for the original economy E if and only if $((1 + \tau_1)\pi_1, \ldots, (1 + \tau_n)\pi_n)$ is an equilibrium for the pure exchange economy E'.

Appendix B: Proof of Proposition 2

Suppose that consumer i has a Cobb-Douglas utility function of the form

$$u_i(x_i) = \prod_{j=1}^{n} x_{ij}^{\alpha_{ij}}$$

where $\alpha_{ij} \geq 0$, $\sum_{j=1}^{n} \alpha_{ij} = 1$. It is known (and easy to see) that the resulting demand is such that consumer i will spend on good j an α_{ij} fraction of her income. In view of the budget constraint (3), the i-th consumer income is $\pi \cdot w_i + \theta_i R$, with "apparent" prices $\pi_j (1 + \tau_{ij})$. Thus,

$$x_{ij}(\pi, R) = \frac{\alpha_{ij}}{\pi_j (1 + \tau_{ij})} (\pi \cdot w_i + \theta_i R),\ j = 1, \ldots, n.$$

We want to derive a consumer's problem that looks like the consumer's problem of an economy without taxes, and that induces the same demand as the original one. The new problem will have an additional good, the "tax good", a utility function $v(\cdot)$ defined on $n + 1$ goods, and a budget constraint as in (4). The demand for the tax good has to be

$$x_{ir}(\pi, R) = \frac{\sum_j \pi_j \tau_{ij} x_{ij}(\pi, R)}{R} = \frac{\sum_j \frac{\alpha_{ij} \tau_{ij}}{1 + \tau_{ij}}}{R} (\pi \cdot w_i + \theta_i R).$$

Observe that this demand looks exactly like the demand generated by a Cobb-Douglas utility function: a constant divided by the price and multiplied by the income. Thus, if we set $\tilde{\alpha}_{ir} = \sum_j \frac{\alpha_{ij} \tau_{ij}}{1 + \tau_{ij}}$, $\tilde{\alpha}_{ij} = \alpha_{ij} / (1 + \tau_{ij})$, the utility function v that we want is the Cobb-Douglas function

$$v(x_{i1}, \ldots, x_{in}, x_{ir}) = x_{ir}^{\tilde{\alpha}_{ir}} \prod_j x_{ij}^{\tilde{\alpha}_{ij}}.$$

Appendix C: Analysis of algorithm RSR

The correctness of the aggregation and disaggregation processes executed in steps 2 and 4 of Algorithm RSR readily follows from Eisenberg's results [12].

Consider now the two-trader problem of step 3, where we approximately compute a zero of B_1 by means of the bisection method, starting from the interval $[\alpha_{min}, \alpha_{max}]$. Each bisection step depends on the sign of B_1 and is thus guided by the solution to an instance of the constrained maximization problem.

The next Lemma gives a stopping criterion which guarantees an ε-equilibrium:

Lemma 5 *If for some α we have $0 \le B_1(\alpha) \le \alpha(1-\alpha)\varepsilon$ then the prices $\pi(\alpha)$ and the allocations $x_1(\alpha)$ and $(1-\varepsilon)x_2(\alpha)$ are an ε-equilibrium.*

Proof : The hypothesis of the Lemma implies that $B_1(\alpha) \le \varepsilon \min\{\alpha, 1-\alpha\}$.

The ε-optimality of the first consumer is guaranteed when $B_1(\alpha) \le \varepsilon\alpha$: since $\alpha > 0$, we have that $0 \le B_1(\alpha) \le \varepsilon\alpha$ is equivalent to $0 \le \pi(\alpha) \cdot z - \pi(\alpha) \cdot x_1(\alpha) \le \alpha\varepsilon$ and, using that $\alpha = \pi(\alpha) \cdot x_1(\alpha)$, it is equivalent to

$$\frac{1}{1+\varepsilon}\pi(\alpha) \cdot z \le \pi(\alpha) \cdot x_1(\alpha) \le \pi(\alpha) \cdot z. \tag{8}$$

On the other hand, if we want a similar inequality for the other trader, we observe that

$$0 \ge B_2(\alpha) \ge -\varepsilon(1-\alpha) \tag{9}$$

is equivalent to

$$\pi(\alpha) \cdot s \le \pi(\alpha) \cdot x_2(\alpha) \le \frac{1}{1-\varepsilon}\pi(\alpha) \cdot s. \tag{10}$$

Thus the bundles $x_1(\alpha)$ and $x_2(\alpha)$ nearly exhaust but do not significantly exceed the respective budgets $\pi(\alpha) \cdot z$ and $\pi(\alpha) \cdot s$. Since they have the "right shape", we are at an $O(\varepsilon)$-equilibrium.

□

Corollary 6 *Suppose that $B_1(\cdot)$ is differentiable in $(0,1)$ and that there exists $M > 0$ such that for all $\alpha \in [\alpha_{min}, \alpha_{max}]$ we have $\frac{d}{d\alpha}B_1(\alpha) \ge -M$. If the bisection method applied to B_1 in $[\alpha_{min}, \alpha_{max}]$ stops when it determines an interval $[\alpha, \beta]$ such that $\beta - \alpha \le \frac{\alpha(1-\alpha)\varepsilon}{M}$ and $B_1(\alpha) \ge 0$, $B_1(\beta) \le 0$, then the conclusion of Lemma 5 follows.*

Proof : Under the hypotheses, $B_1(\alpha) - B_1(\beta) \le \alpha(1-\alpha)\varepsilon$. In particular, $B_1(\alpha) \le \alpha(1-\alpha)\varepsilon$ and the conclusion of Lemma 5 follows. □

Now we need to argue that from an ε-approximate equilibrium to the two trader economy we can recover an ε-approximate equilibrium to the original economy with taxes. We will consider the implications of undoing the aggregations and disaggregations into equivalent economies one by one, in reverse order.

Suppose that the algorithm reaches the conclusion of Lemma 5 at $\alpha = \alpha_1$. Let $\tilde{\pi} = \pi(\alpha_1)$, $\tilde{x}_1 = x_1(\alpha_1)$, $\tilde{x}_2 = x_2(\alpha_1)$, and $\alpha_2 = 1 - \alpha_1$. Write the maximization problem in Step 5 of the algorithm in its extensive form:

$$\begin{aligned}
\max \quad & \alpha_1(\sum_i \gamma_i u_i(x'_i)) + (1-\alpha_1)(\sum_i \theta_i u_i(x''_i)) \\
\text{s.t.} \quad & \sum_i x'_i = x_1 \\
& \sum_i x''_i = x_2 \\
& x_1 + x_2 = w \\
& x'_i, x''_i \geq 0
\end{aligned}$$

Let $\tilde{x}'_i = x'_i(\alpha_1)$ and $\tilde{x}''_i = x''_i(\alpha_1)$. Obviously, we have $\sum_i \tilde{x}'_i = \tilde{x}_1$ and $\sum_i \tilde{x}''_i = \tilde{x}_2$. It can be shown, using the log-homogeneity of the utility functions, that $\tilde{\pi} \cdot \tilde{x}'_i = \gamma_i \alpha_1$ and $\tilde{\pi} \cdot \tilde{x}''_i = \theta_i \alpha_2$.

For the disaggregation into $2m$ traders, we further note that $\tilde{x}'_i$ and $\tilde{x}''_i$ have the right shape – they are proportional to the bundles demanded by the two instances of the i-th trader. Since $\alpha_1 = \tilde{\pi} \cdot \tilde{x}_1$ roughly equals $\tilde{\pi} \cdot z$ (recall the proof of Lemma 5), it follows that $\tilde{\pi} \cdot \tilde{x}'_i = \gamma_i \alpha_1$ roughly equals $\gamma_i \tilde{\pi} \cdot z$. Similarly, $\tilde{\pi} \cdot \tilde{x}''_i = \theta_i \alpha_2$ roughly equals $\theta_i \tilde{\pi} \cdot s$. So we are at an approximate equilibrium of the 2m trader economy.

For the reaggregation of pairs of traders, it is immediate that $\tilde{\pi}$ is an approximate equilibrium of the m trader pure exchange economy. The approximate utility maximizing bundles are $x_i = x'_i + x''_i$.

The natural candidate equilibrium prices for the original economy with taxes are $\pi_j = \tilde{\pi}_j/(1+\tau_j)$. We set $R = \sum_j \pi_j \tau_j w_j$. Since $w_j = \sum_i x_{ij}$, $R = \sum_{ij} \pi_j \tau_j x_{ij}$. Obviously, we have $\sum_i x_i = w$. We can easily verify that the x_i are approximate utility maximizing bundles for each trader at prices π and R.

The running time of Algorithm RSR is dominated by step (3), which can be executed in time proportional to $T(n)(\log M + \log \frac{1}{\varepsilon} + \log \frac{1}{\alpha_{min}} + \log \frac{1}{(1-\alpha_{max})})$, where $T(n)$ is the polynomial bound on the time required to solve the convex program.

Appendix D: Proof of Theorem 4

Invariants. We begin by stating some important invariants maintained by the algorithm.

Proposition 7 *At any stage in the algorithm, we have (1) $y_{ij} > 0 \implies j \in D_i(\pi)$; (2) $h_{ij} > 0 \implies j \in D_i((\frac{\pi_j}{1+\delta}, \pi_{-j}))$, where (r, π_{-j}) is defined to be the vector that differs from π only in its j-th component, which is r; (3) $\pi_j > 1 \implies \sum_i x_{ij} = 1$; (4) The linear program in Allocatemore() is always feasible.*

Proof : The variables y, h, and π, are only changed by calls to Allocatemore() or Raiseprice(). Note that the linear program in Allocatemore() is feasible if (1) and (3) hold: $y'_i = y_i$ and $h'_i = h_i$ yields a feasible solution. On the other hand, it is evident that Allocatemore() preserves (1), (2), and (3).

It is also easy to check that a call to Raiseprice() preserves (1), (2), and (3). Arguments are similar to those in [13]. □

Proposition 8 *At any stage of the algorithm, we have: (1)* $\sum_i x_{ij} \leq 1$ *for each* j*; (2)* $\sum_i T'_i(y_i, h_i, \pi) \leq R$.

Proof : Easily follows by inspecting Allocatemore() and Raiseprice(). □

Proposition 9 *At any stage of the algorithm, except possibly after the execution of the (terminal) assignment statement in Step 1 of Phase 2, we have, for each* i*, that* $\sum_j \pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij} + T'_i(y_i, h_i, \pi) \leq \theta_i R + \sum_j \pi_j w_{ij}$.

Proposition 10 *Let* π_{max} *denote the largest component of* π *and* π_{min} *the smallest component of* π*. At any stage of the algorithm, we have (1)* $\frac{\pi_{max}}{\pi_{min}} \leq \frac{2a_{max}(1+\tau_{max})}{a_{min}}$*; (2)* $R \leq \max\{1, 2n\tau_{max}\pi_{max}\}$.

Proof : To prove (1), it is sufficient to consider the case when $\pi_{max} > 1$. Suppose $\pi_{max} = \pi_j > 1$. Then by statement (3) of Proposition 7, we must have $x_{ij} > 0$ for some i. By Proposition 7, we either have $j \in D_i(\pi)$ or $j \in D_i((\frac{\pi_j}{1+\delta}, \pi_{-j}))$. In either case, it is true that $j \in D_i((\frac{\pi_j}{1+\delta}, \pi_{-j}))$. Thus $\frac{a_{ij}(1+\delta)}{(1+\tau_{ij})\pi_j} \geq \frac{a_{ik}}{(1+\tau_{ik})\pi_k}$ for every k. Rearranging, we have $\frac{\pi_j}{\pi_k} \leq \frac{a_{ij}(1+\tau_{ik})(1+\delta)}{a_{ik}(1+\tau_{ij})}$, which completes the proof of (1).

For (2), observe that if at some stage $R > 1$, then $R = (1+\delta)R'$, where $R' = \sum_i T'_i(\hat{y}_i, \hat{h}_i, \hat{\pi})$, where $\hat{y}_i$,$\hat{h}_i$, and $\hat{\pi}$ are the values at some previous stage of y_i, h_i, and π. Using statement (1) of Proposition 8, we see that

$$\begin{aligned}
\sum_i T'_i(\hat{y}_i, \hat{h}_i, \hat{\pi}) &\leq \sum_i T_i(\hat{x}_i, \hat{\pi}) = \sum_i \sum_j \tau_{ij}\hat{x}_{ij}\hat{\pi}_j \\
&\leq \sum_i \sum_j \tau_{max}\hat{x}_{ij}\hat{\pi}_j = \tau_{max} \sum_j \hat{\pi}_j \sum_i \hat{x}_{ij} \\
&\leq \tau_{max} \sum_j \hat{\pi}_j.
\end{aligned}$$

Thus $R \leq (1+\delta)\tau_{max} \sum_j \hat{\pi}_j \leq (1+\delta)n\tau_{max}\pi_{max}$, since $\pi_{max} \geq \pi_j \geq \hat{\pi}_j$ for every j (the algorithm never decreases a price). This completes the proof. □

Running Time. For each phase, we bound the number of times we call Raiseprice and the number of times we raise the value of R. First observe that the moment π_{min} exceeds 1 in Phase 1, we jump to Phase 2 via Step 4 of phase 1.

Thus $\pi_{min} \leq (1+\delta) \leq 2$ in Phase 1, and by statement (1) in Proposition 10, we have $\pi_{max} \leq \frac{4a_{max}(1+\tau_{max})}{a_{min}}$ in Phase 1. Thus, by statement (2) in Proposition 10, we have $R \leq \max\{1, \frac{8n\tau_{max}(1+\tau_{max})a_{max}}{a_{min}}\} \leq \frac{8n(1+\tau_{max})^2 a_{max}}{a_{min}}$ in Phase 1.

Since R is never raised in Phase 2, we have that

$$R \leq \alpha = \frac{8n(1+\tau_{max})^2 a_{max}}{a_{min}} \tag{11}$$

throughout the algorithm.

We now use this bound to bound π_{max} in Phase 2. Phase 2 stops when $R \leq \delta \sum_j \pi_j w_{ij}$ for each i, due to the stopping rule in Step 1 of Phase 2. To have $R > \delta \sum_j \pi_j w_{ij}$ for some i, we must have $R > \delta \pi_{min} w_{min}$, which implies that $\pi_{min} < \frac{R}{\delta w_{min}}$. Thus we must have $\pi_{min} < \frac{(1+\delta)R}{\delta w_{min}} \leq \frac{2R}{\delta w_{min}}$ throughout Phase 2. Using statement (1) in Proposition 10, and Equation 11, we have $\pi_{max} \leq \frac{16n(1+\tau_{max})^2 a_{max}}{a_{min}\delta w_{min}}$. Since this bound is larger than the bound we established for π_{max} in Phase 1, we conclude that

$$\pi_{max} \leq \beta = \frac{16n(1+\tau_{max})^2 a_{max}}{a_{min}\delta w_{min}} \tag{12}$$

It follows that the number of calls to Raiseprice plus the number of times R is raised is bounded by $n \log_{1+\delta} \alpha + \log_{1+\delta} \beta$, which is a polynomial in the input size, $1/\varepsilon$, and κ. It follows that the running time of the algorithm is bounded by a polynomial in the input size, $1/\varepsilon$, and κ.

Correctness. We now argue that the algorithm terminates at an ε-approximate equilibrium. Suppose that the algorithm terminates either via Step 3 of Phase 1 or Step 4 of Phase 2.

Then, we have for each i, $T_i'(y_i, h_i, \pi) + \sum_j \pi_j x_{ij} \geq T_i'(y_i, h_i, \pi) + \sum_j (\pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij}) = \theta_i R + \sum_j \pi_j w_{ij}$. Adding over all i, we get

$$\sum_i T_i'(y_i, h_i, \pi) + \sum_j \pi_j \sum_i x_{ij} \geq R + \sum_j \pi_j \sum_i w_{ij}. \tag{13}$$

Multiplying the j-th inequality in statement (1) of Proposition 8 by π_j, adding these inequalities and the inequality in statement (2) of Proposition 8, we get

$$\sum_i T_i'(y_i, h_i, \pi) + \sum_j \pi_j \sum_i x_{ij} \leq R + \sum_j \pi_j \sum_i w_{ij}. \tag{14}$$

Thus for Equation 13 to hold, the inequalities in Proposition 8 must be equalities.

So we have for each j that $\sum_i x_{ij} = \sum_i w_{ij}$. Also, we have $\sum_i T_i(x_i, \pi) \leq \sum_i (1+\delta) T_i'(y_i, h_i, \pi) = (1+\delta)R$, whereas $\sum_i T_i(x_i, \pi) \geq \sum_i T_i'(y_i, h_i, \pi) = R$, so we have $R \leq \sum_i T_i(x_i, \pi) \leq (1+\delta)R$.

For each trader i, we have $T_i(x_i, \pi) + \sum_j \pi_j x_{ij} \geq T_i'(y_i, h_i, \pi) + \sum_j \pi_j x_{ij} \geq \theta_i R + \sum_j \pi_j w_{ij}$. Thus the bundle x_i costs i at least her income. But by Proposition 7, we have that if $x_{ij} > 0$ then either $j \in D_i(\pi)$ or $j \in D_i((\frac{\pi_j}{1+\delta}, \pi_{-j}))$. Thus i only spends money on goods that nearly optimize her 'bang for the buck'. It follows via standard arguments that $u_i(x_i) \geq \frac{v_i(\pi, R)}{1+\delta}$.

On the other hand, $T_i(x_i, \pi) + \sum_j \pi_j x_{ij} \leq (1+\delta)(T_i'(y_i, h_i, \pi) + \sum_j (\pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij})) = (1+\delta)(\theta_i R + \sum_j \pi_j w_{ij})$, so the bundle x_i does not cost i more than $1+\delta$ times its income.

Thus we are at a δ-approximate equilibrium.

Now suppose that the algorithm terminates at either Step 1 of Phase 2 or Step 3 of Phase 2. Note that we enter Phase 2 when $\pi_k > 1$ for every good k, so by statement (3) of Proposition 7 we have $\sum_i x_{ik} = 1$ for every good k. Moreover, termination at either of these steps implies that $\sum_i T'(y_i, h_i, \pi_i) = R$, which implies that $R \leq \sum_i T(x_i, \pi) \leq (1+\delta)R$.

We now reason about the budget constraints of the traders. By Proposition 9, we have, for each i

$$T_i(x_i, \pi) + \sum_j \pi_j x_{ij} \leq (1+\delta)(T_i'(y_i, h_i, \pi) + \sum_j \pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij}) \leq (1+\delta)(\theta_i R + \sum_j \pi_j w_{ij}),$$

if we terminate via Step 3 of Phase 2.

If we terminate after making the assignment to R in Step 1 of Phase 2, then we have $R \leq \delta \sum_j \pi_j w_{ij}$ for each i just before the assignment, which implies that $\theta_i R + \sum_j \pi_j w_{ij} \leq (1+\delta)\sum_j \pi_j w_{ij}$ just before the assignment. So the assignment decreases the income of each trader i by at most a multiplicative factor of $(1+\delta)$. Since the inequality in Proposition 9 holds just prior to the assignment, we have that for each i,

$$T_i'(y_i, h_i, \pi) + \sum_j \pi_j y_{ij} + \frac{\pi_j}{1+\delta} h_{ij} \leq (1+\delta)(\theta_i R + \sum_j \pi_j w_{ij})$$

after the assignment.

It follows that on termination, we have for each i

$$T_i(x_i, \pi) + \sum_j \pi_j x_{ij} \leq (1+\delta)^2(\theta_i R + \sum_j \pi_j w_{ij}) \leq (1+3\delta)(\theta_i R + \sum_j \pi_j w_{ij}). \tag{15}$$

Note that this inequality also holds if we terminate via Step 3 of Phase 2.

Having argued that no trader exceeds his budget by much, we now argue that each trader almost exhausts his budget. We have

$$\sum_i (\theta_i R + \sum_j \pi_j w_{ij} - \sum_j \pi_j x_{ij} - T_i(x_i, \pi)) = \sum_j \pi_j (\sum_i w_{ij} - \sum_i x_{ij}) + R - \sum_i T_i(x_i, \pi) \leq 0. \tag{16}$$

It follows that for any trader i', we have

$$\theta_{i'} R + \sum_j \pi_j w_{i'j} - \sum_j \pi_j x_{i'j} - T_{i'}(x_{i'}, \pi) \leq \sum_i \max\{0, \sum_j \pi_j x_{ij} + T_i(x_i, \pi) - \theta_i R - \sum_j \pi_j w_{ij}\}.$$

Using Equation 15, and Proposition 10, we have

$$\begin{aligned}
\theta_{i'} R + \sum_j \pi_j w_{i'j} - \sum_j \pi_j x_{i'j} - T_{i'}(x_{i'}, \pi) &\leq \sum_i 3\delta(\theta_i R + \sum_j \pi_j w_{ij}) = 3\delta(R + \sum_j \pi_j) \\
&\leq 3\delta(2n\pi_{max} + 2n\pi_{max}\tau_{max} + \sum_j \pi_j) \\
&\leq 3\delta 3n\pi_{max}(1+\tau_{max}) \\
&\leq \frac{\varepsilon}{10}\pi_k w_{i'k} \leq \frac{\varepsilon}{10}(\sum_j \pi_j w_{i'j} + \theta_{i'} R).
\end{aligned}$$

Here k is the good for which $\pi_{max} = \pi_k$.

We have argued that each trader i spends at least a fraction $\frac{1}{1+\varepsilon}$ of her budget on bundle x_i. But by Proposition 7, we have that if $x_{ij} > 0$ then either $j \in D_i(\pi)$ or $j \in D_i((\frac{\pi_j}{1+\delta}, \pi_{-j}))$. Thus i only spends money on goods that nearly optimize her bang for the buck. It follows that $u_i(x_i) \geq (1 - c\varepsilon)v_i(\pi, R)$, where $c \geq 1$ is some constant. Thus we are at a $c\varepsilon$-approximate equilibrium.