Random monomial ideals

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Why?

A probability distribution on ideals allows us to study the distribution on algebraic properties such as Hilbert function, regularity, projective dimension, etc. We thus formalize "typical" behavior in commutative algebra.

Monomial ideals generalize graphs and simplicial complexes. Our inspiration is the Erdős–Rényi random graph model $G(n, p)$. A random graph on $n$ vertices is constructed by including each of the possible $\binom{n}{2}$ edges with (independent) probability $p$.

The Erdős–Rényi-type model

Let $S = k[x_1, \ldots, x_n]$, with $k$ a field.

**Parameters:** $n$, number of variables

$D$, a fixed maximum degree

$p$, a parameter $0 \leq p \leq 1$

**Procedure:**

- Select a random generating set $\mathfrak{B}$: for each monomial $x^\alpha \in k[x_1, \ldots, x_n]$ with $1 \leq |\alpha| \leq D$, include $x^\alpha$ in $\mathfrak{B}$ with (independent) probability $p$.
- Define the random monomial ideal by $\mathfrak{I} = (\mathfrak{B})$

**Notation:** $\mathfrak{B} \sim B(n, D, p)$ means $\mathfrak{B}$ is sampled from the distribution on generating sets. $\mathfrak{I} \sim I(n, D, p)$ means $\mathfrak{I}$ is sampled from the resulting distribution on ideals.

Threshold functions

Thresholds describe sudden transitions in behavior as the model parameters vary. In the limit the transition is a step function.

![Classic $G(n,p)$ threshold vs An example in our model](image)

What is the distribution $I(n, D, p)$?

**Theorem 1.** Let $I \subseteq S$ be a fixed monomial ideal with Hilbert function $H_I(d)$ and number of minimal generators $\beta_1$. Then for $\mathfrak{I} \sim I(n, D, p)$,

$$P[\mathfrak{I} = I] = p^{\beta_1}(1 - p)^{\sum_{d=1}^{D} H_I(d)}.$$

Distribution on Hilbert functions

The probability of generating a random monomial ideal with a given Hilbert function $h$ follows from Theorem 1 and the number of vertices of the 0-1 polytopes encoding the distinct monomial ideals in $n$ variables, with generators of degree less than or equal to $D$, Hilbert function $h$, and specified first graded Betti numbers.

Krull dimension of $S/\mathfrak{I}$

**Theorem 2.** Let $\mathfrak{I} \sim I(n, D, p)$, and let $t$ be an integer, $0 \leq t \leq n$. Then the probability that $S/\mathfrak{I}$ has Krull dimension $t$ is a polynomial in $p$ of degree $\sum_{i=1}^{t} (\binom{n}{i})$.

**Ex.** $P[\dim S/\mathfrak{I} = 0] = (1 - (1 - p)^D)^n$.

**Theorem 3.** Let $n$ be fixed, and $p = p(D)$. Suppose $\mathfrak{I} \sim I(n, D, p)$. For any integer $t$, $1 \leq t \leq n$, $D^t$ is a threshold function for the property that $\dim(S/\mathfrak{I}) \leq t - 1$. That is,

$$\lim_{D \to \infty} P[\dim(S/\mathfrak{I}) \leq t - 1] = \begin{cases} 0, & \text{if } p = o(D^{-t}) \\ 1, & \text{if } p = \omega(D^{-t}) \end{cases}.$$

**Ex.** For $D$ sufficiently large, $\mathfrak{I} \sim I(10, D, D^{-3.5})$ generates 3-dimensional ideals in 10 variables with high probability.

Degrees of minimal generators

The choice of $p$ yields the minimum degree of a minimal generator for $\mathfrak{I} \sim I(n, D, p)$.

**Theorem 4.** (a) If $D$ is fixed, $d \leq D$ a constant, and $p = p(n)$. Then $n^{-d}$ is a threshold function for the property that $\mathfrak{I}$ is generated in degrees larger than $d$.

(b) Let $n$ be fixed, $d = d(D) \leq D$ such that $\lim_{D \to \infty} d(D) = \infty$, and $p = p(D)$. Then $d^{-n}$ is a threshold function for the property that $\mathfrak{I}$ is generated in degrees larger than $d$.

and $\reg_0(\mathfrak{I})$, the maximum degree of a minimal generator:

**Theorem 5.** Let $n$ be fixed, $p = p(D)$, and $r = r(D)$ a function tending to infinity as $D \to \infty$. If $p = \omega(\frac{1}{D})$, then $\reg_0(\mathfrak{I}) \leq nr$ a.a.s.

**Ex.** For fixed $n$, if $p(D) = \omega(1/\log D)$, then w.h.p. $\mathfrak{I} \sim I(n, D, p)$ will be generated in degrees no more than $n \log D$.

Connection with random topology

The ER model is an instance of our general model: to each monomial $x^\alpha \in k[x_1, \ldots, x_n]$ with $1 \leq |\alpha| \leq D$, assign a probability $0 \leq p_\alpha \leq 1$. Then sample generating set $\mathfrak{B}$ via $P[x^\alpha \in \mathfrak{B}] = p_\alpha$, and set $\mathfrak{I} = (\mathfrak{B})$.

**Theorem 6.** Given parameters $(p_0, \ldots, p_\alpha)$ in the Costa-Farber random simplicial complex model, set $D = d + 1$ and define

$$p_n = \begin{cases} 1 - p_{|\alpha|-1}, & i \not\in \{0, 1\}^n \\ 0, & \text{otherwise}. \end{cases}$$

Then for any simplicial complex $\Delta$, its probability in the Costa-Farber model is exactly the probability of the Stanley-Reisner ideal $I_\Delta$ in the general model.

As a result our model generalizes $G(n, p)$ and all well-known random simplicial complex models.

Linial-Meshulam

Clique complexes (Kahle)

A conjecture on projective dimension

We implemented the ER model in Macaulay2 to study some algebraic properties experimentally.

**Conjecture:** almost all ideals have projective dimension $n$ as $n$ and $D$ go to infinity, for $p = \omega(n^{-D})$ and $p = \omega(D^{-n})$, respectively.