# Symplectic Fillings and Lefschetz Fibrations

Laura Starkston

December 20, 2015

There is a correspondence between Stein fillings and Lefschetz fibrations which allows us to understand geometric structures (complex/symplectic) through topological decompositions. On the boundary we see the Giroux correspondence between contact structures and open book decompositions. In this lecture, we will explain these notions and their relations to each other.

## 1 Stein and Symplectic Fillings

**Definition 1.1.** A symplectic structure on a manifold $X$ is a *closed*, *non-degenerate* 2-form $\omega$.

Symplectic structures on exist on *even dimensional* manifolds.

In particular, if $(W, \omega)$ is a symplectic manifold with boundary $\partial W = Y$, $\omega|_Y$ cannot be symplectic. At most $\omega|_Y$ can be non-degenerate on a hyperplane field.

**Definition 1.2.** A *contact structure* on a manifold $Y$ is a hyperplane field $\xi$ defined by the kernel of a 1-form $\alpha$ such that $\alpha \wedge d\alpha \neq 0$. Such an $\alpha$ is called a *contact form*.

We say that $(W, \omega)$ has *contact type boundary* if there is a contact form $\alpha$ where $\omega|_{\partial W} = d\alpha$.

A contact structure is a good structure to keep track of on the boundary of a symplectic manifold because it provides enough information to tell us when two symplectic manifolds with boundary will glue together. Gluing requires a model collared neighborhood. There are two types of contact boundary. If $(W, \omega)$ has $\omega|_{\partial W} = d\alpha$, and the orientation induced by the non-vanishing form $\alpha \wedge d\alpha$ agrees with the boundary orientation we say $(W, \omega)$ has *convex* boundary. If the two orientations disagree we say $(W, \omega)$ has *concave* boundary.

If $(W, \omega)$ has convex boundary, we can find a collared neighborhood symplectomorphic to $([-\varepsilon, 0] \times \partial W, d(e^t\alpha))$ identifying $\partial W$ with $\partial W \times \{0\}$. Similarly the collar model for concave boundary is $([0, \varepsilon] \times \partial W, d(e^t\alpha))$. (The $t$ coordinate is in the interval.)

Notice that the vector field $\partial_t$ satisfies the equation $\mathcal{L}_{\partial_t}\omega = \omega$ for $\omega = d(e^t\alpha)$ because by Cartan's formula

$$\mathcal{L}_{\partial_t}(d(e^t\alpha)) = \iota_{\partial_t}d(d(e^t\alpha)) + d(\iota_{\partial_t}d(e^t\alpha)) = d(\iota_{\partial_t}(e^t(dt \wedge \alpha + d\alpha))) = d(e^t\alpha)$$

Any such vector field $V$ with $\mathcal{L}_V\omega = \omega$ is called a *Liouville vector field.* An everywhere transverse Liouville vector field identifies the boundary with one of these model collars, thus showing the boundary has contact type. The contact form is defined by $\alpha = i^*(\iota_V\omega)$. If the Liouville vector field points outward from the boundary the boundary is *convex*, and if it points inwards the boundary is *concave.*

**Definition 1.3.** A symplectic manifold $(W, \omega)$ with convex contact type boundary is called a *(strong) symplectic filling* of the induced contact boundary. A symplectic manifold $(W, \omega)$ with concave contact type boundary is called a *concave cap* of the induced contact boundary.

There is a significant asymmetry between symplectic fillings and concave caps. For example, every contact 3-manifold has a concave cap (usually many concave caps with varying topological properties), but certain classes of contact manifolds have no symplectic fillings. This makes constructions of symplectic fillings particularly important.

One source of symplectic fillings comes from complex geometry:

**Definition 1.4.** A *Stein manifold* is a complex manifold which admits a proper complex embedding into $\mathbb{C}^N$.

Such manifolds are necessarily non-compact, but by intersecting with a large ball in $\mathbb{C}^N$, we obtain a compact manifold with boundary called a *Stein domain*. A Stein domain inherits a symplectic structure from the standard symplectic structure $\omega_{std} = \sum_i dx_i \wedge dy_i$ on $\mathbb{C}^N$. Restricting the radial vector field $\sum_i \frac{1}{2}(x_i\partial_{x_i} + y_i\partial_{y_i})$ to the Stein domain, provides a Liouville vector field which points outwardly transverse to the boundary. Therefore a Stein domain has convex contact type boundary.

We say the Stein domain is a *Stein filling* of its induced contact boundary.

While every Stein filling is a symplectic filling, the converse is not true. The first examples of (strong) symplectic fillings which are not Stein are due to Ghiggini. However, there are certain kinds of contact manifolds where every strong symplectic filling is deformation equivalent to a Stein filling.

## 2 Lefschetz fibrations

Now we look at the topological side.

**Definition 2.1.** A Lefschetz fibration on a 4-manifold $X$ is a map $\pi : X \to D^2$ which has finitely many critical values $t_1, \cdots, t_n \in \mathring{D}^2$ such that there is a unique critical point $d_i \in \pi^{-1}(d_i)$ mapping to each critical value, and near each critical point we can choose complex coordinates $(z_1, z_2)$ such that in these coordinates $\pi(z_1, z_2) = z_1^2 + z_2^2$.

Away from the critical values, $\pi$ is a fibration. When $X$ is a 4-manifold with boundary, the generic fiber $F$ is a surface (with boundary). If the Lefschetz fibration has no critical points, then $X \cong F \times D^2$. Introducing critical points changes the topology but in a carefully controlled way. Let $U$ be a neighborhood of a critical point with complex coordinates $(z_1, z_2)$ so that $\pi(z_1, z_2) = z_1^2 + z_2^2$ in $U$ and the critical point is at $(0, 0)$. Writing $z_j = x_j + iy_j$ we have

$$\pi(x_1 + iy_1, x_2 + iy_2) = x_1^2 + x_2^2 - y_1^2 - y_2^2 + i(2x_1y_1 + 2x_2y_2)$$

In this explicit form, we can see what happens to the fibers as we approach the critical value. For simplicity we will look at the family of fibers above positive real numbers approaching 0. Above $c \in \mathbb{R}_+$ we have

$$\pi^{-1}(c) \cap U = \{(x_1 + iy_1, x_2 + iy_2) : x_1^2 + x_2^2 - y_1^2 - y_2^2 = c, x_1y_1 + x_2y_2 = 0\}$$

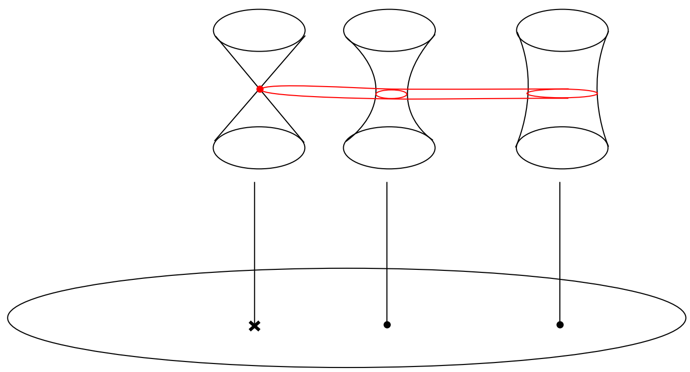

Figure 1: Fibers near a Lefschetz singularity. Vanishing cycle collapses to a point.

With some observation, we see this fiber is an annulus with core circle $C_c = \{x_1^2 + x_2^2 = c, y_1 = y_2 = 0\}$. The rest of the annulus is made up of the pair of circles which solve the equations $\{x_1^2 + x_2^2 = c + r, < x_1, x_2 > \cdot < y_1, y_2 >= 0, y_1^2 + y_2^2 = r\}$ for $r > 0$. (Once the values $x_1, x_2$ are chosen on the circle of radius $c + r$, there are exactly two choices for the pair $(y_1, y_2)$ on the circle of radius $r$ which represent a vector orthogonal to $< x_1, x_2 >\in \mathbb{R}^2$.) As $c \to 0$ the core circle of the annulus shrinks to a point.

The circle $C_c$ in the non-singular fiber is called a *vanishing cycle*. The collection of circles $C_c$ together with the origin form a disk called the *thimble*.

The difference between a neighborhood of this singular fiber and the neighborhood of a regular fiber is a single 2-handle attached along the vanishing cycle. (Take the Morse function given by $f = -Re(\pi) = y_1^2 + y_2^2 - x_1^2 - x_2^2$ to see the Morse coordinates near the index 2 critical point.) Calculating the framing is a little more tricky, but by looking at a coordinate frame for the fiber near the boundary of the thimble, one can calculate that the framing for the 2-handle is $fr(F) - 1$ where $fr(F)$ is the framing of the vanishing cycle induced by the fiber $F$ i.e. $fr(F)$ is a non-zero section of the normal bundle to the curve in the surface $F$.

In the 4-manifold $X$, we just saw that a neighborhood of a Lefschetz critical fiber is built by attaching a 2-handle to the trivial bundle $F \times D^2$. By gluing together a bunch of these neighborhoods of singular fibers along an interval of regular fibers $F$ along their boundary, we can build a Lefschetz fibration with any finite number of critical points with specified vanishing cycles.

## 3 On the boundary: Open book decomposition

The boundary of a Lefschetz fibration $\pi : X \to D^2$ naturally splits into two pieces:

The *vertical boundary* is made up of the union of the fibers $F$ over points $x \in \partial D^2$. Because there are no critical values in $\partial D^2$, this is a fibration over $\partial D^2 \cong S^1$ with fiber $F$.

The *horizontal boundary* is made up of the union of the boundaries of all of the fibers. Each fiber has boundary diffeomorphic to $\sqcup_n S^1$ and there is a $D^2$ family worth of these. Thus the horizontal boundary is diffeomorphic to a disjoint union of solid tori $\sqcup_n S^1 \times D^2$.

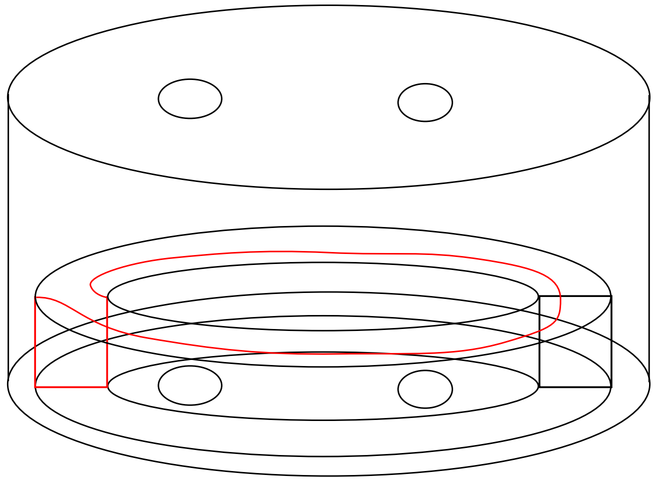

Figure 2: Shows how $fr(F) - 1$ surgery on $F \times S^1$ gives rise to a manifold $F \times [0,1]/(x,1) \sim (\phi(x), 0)$ where $\phi$ is a right-handed Dehn twist.

Note that every point in $\partial X$ lies in at least one of these pieces and the overlap is a disjoint union of tori: $\partial F \times \partial D \cong \sqcup S^1 \times S^1$.

The fibers $F$ of the vertical boundary are called the *pages* of the open book decomposition and the central circles $S^1 \times \{0\}$ of the horizontal boundary are called the *binding*.

This type of decomposition is precisely the form of an *open book decomposition*. The data needed to abstractly build a manifold diffeomorphic to $\partial X$ using the open book decomposition is the diffeomorphism type of the fiber $F$ together with the *monodromy* $\phi$ of the fibration over $S^1$. Then one can rebuild a manifold diffeomorphic to $\partial X$ by taking $F \times [0,1]$, and gluing the two ends together by $\phi$ and then collapsing the intervals on the boundary:

$$\partial X \cong F \times [0,1]/\sim$$

where $(1,x) \sim (0, \phi(x))$ for all $x \in F$ and $(t,x) \sim (t',x)$ for all $x \in \partial F$.

Each Lefschetz critical point adds a 2-handle attached along a vanishing cycle in a fiber $F$ over a point in $\partial D^2$ with framing $fr(F) - 1$. The change in the boundary is to perform Dehn surgery along that vanishing cycle with framing $fr(F) - 1$. Figure 2 shows how this Dehn surgery changes the monodromy of the open book by adding a positive Dehn twist into the monodromy.

Open book decompositions exist on every 3-manifold. They were used as a way to create contact structures by the Thurston-Winkelnkemper construction: take a 2-plane field tangent to the pages except near the boundary (binding) where it rotates to become positively transverse to the binding, then perturb this plane field a small amount to make it a contact structure. The converse is the Giroux correspondence which shows that in fact every contact structure has a corresponding open book decomposition.

# 4 Connecting Lefschetz fibrations with Stein fillings

A Lefschetz fibration on a 4-manifold gives rise to a Stein structure (recall this is a special case of a symplectic filling). The key to this is the following theorem of Eliashberg

**Theorem 4.1** (Eliashberg)**.** *A* $2n$*-manifold with a handle decomposition with handles of index* $\leq n$ *admits a Stein structure if*

- $n \neq 2$ *(i.e. not a* 4*-manifold)*

  *OR*

- $n = 2$ *and the* 2*-handles are attached along Legendrian knots in the contact boundary of the* 0*- and* 1*-handles with framing* $tb - 1$*.*

*Conversely, any Stein manifold admits such a handle decomposition.*

In dimension 4, Gompf developed a way to present Kirby calculus diagrams encoding the Stein structure from such a handle decomposition.

We have a Stein structure on $F \times D^2$ for any surface $F$ with boundary. The contact structure on the boundary is isotopic to the one induced by the trivial open book which on $F \times S^1$ is almost tangent to the fibers $F$. Therefore for a circle in a fiber $F$, the framings agree $fr(F) - 1 = tb - 1$, so we can extend the Stein structure over the 2-handles by the above theorem.

Conversely, given a Stein filling it is possible to find a corresponding Lefschetz fibration. This was shown first by Loi-Piergallini, and then by a more explicit construction by Akbulut-Ozbagci. Both methods use Eliashberg's characterization of Stein manifolds in terms of handle-decompositions. Therefore Stein fillings (up to deformation) are in one to one correspondence with Lefschetz fibrations (up to stabilizations - which increases the topology of the fiber and then cancels the contribution with an extra vanishing cycle, and Hurwitz moves - which exchange the ordering of the vanishing cycles)

If we fix a contact manifold and want to understand all of its Stein fillings, we can look at all the compatible open books and for each one, look at all factorizations of the monodromy into right-handed Dehn twists. In the special case that the contact structure is compatible with a *planar* open book (the pages are genus 0), a theorem of Wendl says that all Stein fillings of that contact manifold are in one to one correspondence with the Lefschetz fibrations whose boundary is that particular planar open book. Therefore Stein fillings are in one to one correspondence with factorizations of that particular monodromy element into right-handed Dehn twists (up to Hurwitz moves).

When we have two *positive factorizations* (factorizations into right-handed Dehn twists) of the same monodromy element, we get a *monodromy substitution*. This corresponds to cutting out one Stein filling corresponding to one of the Lefschetz fibrations and replacing it with another Stein filling corresponding to the other Lefschetz fibration. Often these Stein fillings have different topology. The number of Dehn twists determines the Euler characteristic of the filling, and knowledge of the specific vanishing cycles allows one to compute the homology, intersection form, and fundamental group.