4.1 - (15e) $f: \mathbb{Z}_6 \to \mathbb{Z}_6$ WITH $f(\overline{x}) = [x^2+3]$ IS A WELL-DEFINED FUNCTION.

PF LET $\overline{x} = \overline{y}$ IN $\mathbb{Z}_6$, SO $x \equiv y \pmod 6$ AND THEREFORE $6 \mid (x-y)$.

THEN $x - y = 6k$ FOR SOME $k \in \mathbb{Z}$, SO

$(x^2+3) - (y^2+3) = x^2 - y^2 = (x-y)(x+y) = 6k(x+y)$ WHERE $k(x+y) \in \mathbb{Z}$.

THEREFORE $6 \mid ((x^2+3)-(y^2+3))$, SO $x^2+3 \equiv y^2+3 \pmod 6$ AND

THUS $f(\overline{x}) = [x^2+3] = [y^2+3] = f(\overline{y})$.

CONSEQUENTLY f IS A WELL-DEFINED FUNCTION.

4.2 - (17c) IF f AND g ARE DECREASING ON AN INTERVAL I AND $f \circ g$ IS DEFINED ON I, THEN $f \circ g$ IS INCREASING ON I.

PF LET $x, y \in I$ WITH $x < y$. SINCE g IS DECREASING ON I, $g(x) > g(y)$ SO $g(y) < g(x)$. SINCE f IS DECREASING ON I, $f(g(y)) > f(g(x))$ SO $(f \circ g)(x) < (f \circ g)(y)$. THEREFORE $f \circ g$ IS INCREASING ON I.

REMARK HERE WE ARE ASSUMING THAT $Dom(f) = I$, SO $Rng(g) \subseteq I$.
(SEE ALSO THE BOTTOM OF P. 4.)

4.3 - (1) h) $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ WITH $f(x,y) = x - y$.

f IS ONTO: IF $z \in \mathbb{R}$, THEN $f(z,0) = z$ SO $z \in Rng(f)$.
THEREFORE $Rng(f) = \mathbb{R}$.

j) $f: \mathbb{R} \to [1, \infty)$ WITH $f(x) = x^2 + 1$.

f IS ONTO:

LET $y \in [1,\infty)$, SO $y \geq 1$ AND $y - 1 \geq 0$.

IF $x = \sqrt{y-1}$, THEN $f(x) = (y-1)+1 = y$ SO $y \in Rng(f)$.

THUS $Rng(f) = [1, \infty)$.

e) $f: (1,\infty) \to (1,\infty)$ WITH $f(x) = \dfrac{x}{x-1}$.

f IS ONTO:

LET $y \in (1,\infty)$, AND LET $x = \dfrac{y}{y-1}$.

SINCE $y > y-1$ AND $y - 1 > 0$, $x = \dfrac{y}{y-1} > 1$ SO $x \in (1,\infty)$

AND $f(x) = \dfrac{x}{x-1} = \dfrac{\frac{y}{y-1}}{\frac{y}{y-1} - 1} = \dfrac{y}{y-(y-1)} = \dfrac{y}{1} = y$.

THEREFORE $y \in Rng(f)$, SO $Rng(f) = (1,\infty)$.

(2) j) f IS NOT 1-1, SINCE $f(-1) = 2 = f(1)$.

e) f IS 1-1, SINCE $f(c) = f(d) \Rightarrow \dfrac{c}{c-1} = \dfrac{d}{d-1} \Rightarrow$

$c(d-1) = (c-1)d \Rightarrow cd - c = cd - d \Rightarrow -c = -d \Rightarrow c = d$.

(OR SHOW THAT $f'(x) < 0$ ON $(1,\infty)$, SO f IS DECREASING ON $(1,\infty)$.)

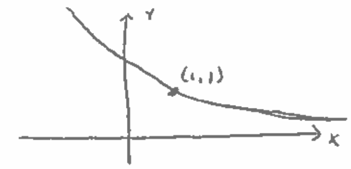
(5) If $f: A \to B$ and $g: B \to C$ are both onto, then $g \circ f: A \to C$ is onto.

PF Let $y \in C$. Since g is onto, $y = g(b)$ for some $b \in B$; and since f is onto, $b = f(a)$ for some $a \in A$.
Therefore $y = g(b) = g(f(a)) = (g \circ f)(a)$, so $y \in \mathrm{Rng}(g \circ f)$.
Thus $\mathrm{Rng}(g \circ f) = C$, so $g \circ f$ is onto.

(6) Let $f: A \to B$ and $g: B \to C$. If $g \circ f: A \to C$ is 1-1, then f is 1-1.

PF Let $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$, so $(g \circ f)(x) = (g \circ f)(y)$.
Since $g \circ f$ is 1-1, we have that $x = y$. Therefore f is 1-1.

(10a) Let $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = \begin{cases} 2 - x, & \text{if } x \le 1 \\ 1/x, & \text{if } x > 1. \end{cases}$



1) f is 1-1.

PF If $x \le 1$, then $f(x) = 2 - x \ge 1$; so $f(x) < 1 \Rightarrow x > 1$. (using the
If $x > 1$, then $f(x) = \frac{1}{x} < 1$; so $f(x) \ge 1 \Rightarrow x \le 1$. contrapositive)
Let $f(x) = f(y)$.
a) If $f(x) \ge 1$, then $x \le 1$ and $y \le 1$ so $2 - x = 2 - y \Rightarrow x = y$.
b) If $f(x) < 1$, then $x > 1$ and $y > 1$ so $\frac{1}{x} = \frac{1}{y} \Rightarrow x = y$.
Since $x = y$ in either case, f is 1-1.

OR PF Let $f(x) = f(y)$.
a) If $x \le 1$ and $y \le 1$, then $2 - x = 2 - y$ so $x = y$.
b) If $x > 1$ and $y > 1$, then $\frac{1}{x} = \frac{1}{y}$ so $x = y$.
c) If $x \le 1$ and $y > 1$, then $f(x) = 2 - x \ge 1$ and $f(y) = \frac{1}{y} < 1$;
so $f(x) \ne f(y)$ and we have a contradiction.
d) If $x > 1$ and $y \le 1$, we also have a contradiction
(interchanging x and y in the previous case).
Therefore $f(x) = f(y) \Rightarrow x = y$, so f is 1-1.

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2) f is not onto $\mathbb{R}$.

PF Suppose $f(x) = 0$.
If $x \le 1$, then $2 - x = 0$ so $x = 2$, which gives a contradiction.
If $x > 1$, then $\frac{1}{x} = 0$ so $1 = 0$, which also gives a contradiction.
Therefore $0 \notin \mathrm{Rng}(f)$, so f is not onto $\mathbb{R}$.

4.3 – 10b $f:\mathbb{R}\to\mathbb{R}$ WITH $f(x)=\begin{cases} x+4, & \text{IF } x\le -2 \\ -x, & \text{IF } -2<x<2 \\ x-4, & \text{IF } x\ge 2. \end{cases}$

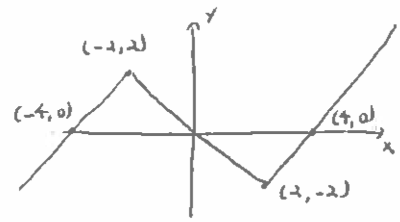
1) f IS ONTO $\mathbb{R}$

PF LET $y\in\mathbb{R}$.

a) IF $y<0$, LET $x=y-4$. THEN $x\le -2$, SO $f(x)=x+4=y$ AND THEREFORE $y\in Rng(f)$.

b) IF $y\ge 0$, LET $x=y+4$. THEN $x\ge 2$, SO $f(x)=x-4=y$ AND THEREFORE $y\in Rng(f)$.

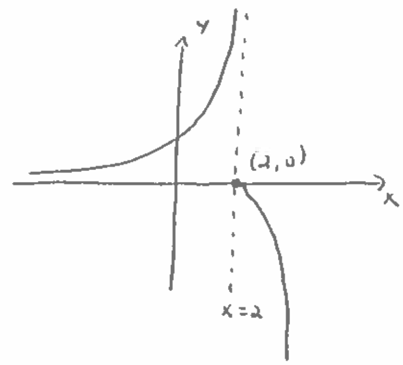
THUS $Rng(f)=\mathbb{R}$, SO f IS ONTO $\mathbb{R}$.



2) f IS NOT 1-1.

PF SINCE $f(0)=0=f(4)$, f IS NOT 1-1.

D.S. 4 – 3 $f:\mathbb{R}\to\mathbb{R}$ WITH $f(x)=\begin{cases} \frac{1}{2-x}, & \text{IF } x<2 \\ 4-x^2, & \text{IF } x\ge 2. \end{cases}$



A) 1) f IS 1-1.

PF IF $x<2$, THEN $2-x>0$ SO $f(x)=\frac{1}{2-x}>0$.

IF $x\ge 2$, THEN $x^2\ge 4$ SO $f(x)=4-x^2\le 0$.

THEREFORE $f(x)\le 0 \Rightarrow x\ge 2$,

AND $f(x)>0 \Rightarrow x<2$.

LET $f(x)=f(y)$.

a) IF $f(x)\le 0$, THEN $x\ge 2$ AND $y\ge 2$ SO $4-x^2=4-y^2$ AND THEREFORE $y^2=x^2$; SO $x=y$ (SINCE $x,y>0$).

b) IF $f(x)>0$, THEN $x<2$ AND $y<2$ SO $\frac{1}{2-x}=\frac{1}{2-y} \Rightarrow 2-x=2-y \Rightarrow x=y$.

SINCE $f(x)=f(y)\Rightarrow x=y$, f IS 1-1.

2) f IS ONTO.

PF LET $y\in\mathbb{R}$.

a) IF $y>0$, LET $x=2-\frac{1}{y}$. THEN $x<2$, SO $f(x)=\frac{1}{2-x}=\frac{1}{1/y}=y$ AND THEREFORE $y\in Rng(f)$.

b) IF $y\le 0$, LET $x=\sqrt{4-y}$. SINCE $4-y\ge 4$, $x\ge 2$; SO $f(x)=4-x^2=4-(4-y)=y$ AND THEREFORE $y\in Rng(f)$.

THUS $Rng(f)=\mathbb{R}$, SO f IS ONTO.

B) LET $y=f(x)$. BY PART A),

a) IF $y>0$, THEN $x=2-\frac{1}{y}$ AND

b) IF $y\le 0$, THEN $x=\sqrt{4-y}$;

SO $f^{-1}(y)=\begin{cases} \sqrt{4-y}, & \text{IF } y\le 0 \\ 2-\frac{1}{y}, & \text{IF } y>0. \end{cases}$

(4) LET $f: \mathbb{R}-\{2\} \to \mathbb{R}-\{5\}$ WITH $f(x) = \dfrac{5x+1}{x-2}$.

1) f IS 1-1

PF IF $f(x)=f(y)$, THEN $\dfrac{5x+1}{x-2} = \dfrac{5y+1}{y-2} \Rightarrow (5x+1)(y-2) = (5y+1)(x-2)$

$\Rightarrow 5xy + y - 10x - 2 = 5xy + x - 10y - 2 \Rightarrow 11y = 11x \Rightarrow x = y$;

SO f IS 1-1.

2) f IS ONTO

PF LET $y \in \mathbb{R}-\{5\}$, AND LET $x = \dfrac{2y+1}{y-5}$. THEN $x = 2 + \dfrac{11}{y-5}$, SO $x \neq 2$;

AND $f(x) = \dfrac{5x+1}{x-2} = \dfrac{5\left(\frac{2y+1}{y-5}\right)+1}{\frac{11}{y-5}} = \dfrac{10y+5+y-5}{11} = \dfrac{11y}{11} = y$.

THEREFORE $y \in Rng(f)$, SO f IS ONTO.

(8) LET $f: \mathbb{R} \to \mathbb{R}$ WITH $f(x) = x^3 - 3x^2 + 4x + 9$.

1) f IS 1-1.

PF SINCE $f'(x) = 3x^2 - 6x + 4 = 3(x^2-2x+1)+1 = 3(x-1)^2+1 > 0$ FOR ALL x,

f IS INCREASING ON $\mathbb{R}$ AND THEREFORE f IS 1-1.

2) f IS ONTO.

PF LET $y \in \mathbb{R}$.

SINCE $\lim_{x\to\infty} f(x) = \infty$, THERE IS A $b \in \mathbb{R}$ WITH $f(b) > y$.

SINCE $\lim_{x\to-\infty} f(x) = -\infty$, THERE IS AN $a \in \mathbb{R}$ WITH $a < b$ AND $f(a) < y$.

SINCE f IS CONTINUOUS ON $[a,b]$, BY THE INTERMEDIATE VALUE TH.

THERE IS A $c \in (a,b)$ WITH $f(c) = y$.

THEREFORE $y \in Rng(f)$, SO $Rng(f) = \mathbb{R}$.

4.2 – (17c) NOTICE THAT IN THE CASE WHERE $f: \mathbb{R} \to \mathbb{R}$ AND $g: \mathbb{R} \to \mathbb{R}$
AND f AND g ARE BOTH DECREASING ON AN INTERVAL I,
IT IS NOT NECESSARILY TRUE THAT $f \circ g$ IS INCREASING ON I;
FOR EXAMPLE, $f(x) = 1 - x^2$ AND $g(x) = -x$ ARE BOTH
DECREASING ON $[0,\infty)$; BUT $f \circ g = f$ IS ALSO DECREASING ON $[0,\infty)$.
