MATH 108 SOLUTIONS – SEC. 4.3 AND 4.4

4.3 – (15b) F – THE CLAIM IS FALSE. (THE "PROOF" DID NOT SHOW THAT $x \in [1,\infty)$.)

4.4 – (1) a) $f: \mathbb{R} \to \mathbb{R}$ WITH $f(x) = mx + b$ $(m \neq 0)$.

1) f IS 1-1.

PF IF $f(x) = f(y)$, THEN $mx + b = my + b \Rightarrow mx = my \Rightarrow x = y$.

2) f IS ONTO.

PF LET $y \in \mathbb{R}$, AND LET $x = \dfrac{y-b}{m}$. THEN $f(x) = mx + b = (y-b) + b = y$,

SO $y \in Rng(f)$. THEREFORE $Rng(f) = \mathbb{R}$.

b) $f: (2,\infty) \to (-\infty,-1)$ WITH $f(x) = \dfrac{-x}{x-2} = \dfrac{x}{2-x}$

1) f IS 1-1.

PF IF $f(x) = f(y)$, THEN $\dfrac{x}{2-x} = \dfrac{y}{2-y} \Rightarrow x(2-y) = y(2-x) \Rightarrow$

$2x - xy = 2y - xy \Rightarrow 2x = 2y \Rightarrow x = y$.

2) f IS ONTO.

PF LET $y \in (-\infty,-1)$, SO $1+y<0$; AND LET $x = \dfrac{2y}{1+y}$.

THEN $x \in (2,\infty)$ SINCE $x - 2 = \dfrac{-2}{1+y} > 0$,

AND $f(x) = \dfrac{x}{2-x} = \dfrac{\frac{2y}{1+y}}{\frac{2}{1+y}} = \dfrac{2y}{2} = y$.

THEREFORE $y \in Rng(f)$, SO $Rng(f) = (-\infty,-1)$.

(2) b) DEFINE $f: \mathbb{N} \to \mathbb{N} - \{1\}$ BY $f(n) = n+1$.

THEN f IS 1-1 SINCE $f(m) = f(n) \Rightarrow m+1 = n+1 \Rightarrow m = n$, AND

f IS ONTO SINCE $m \in \mathbb{N} - \{1\} \Rightarrow m-1 \in \mathbb{N}$ WITH $f(m-1) = m$.

d) DEFINE $f: (-\infty,1) \to (-1,\infty)$ BY $f(x) = -x$.

(NOTICE THAT IF $x < 1$, THEN $-x > -1$ SO $f(x) > -1$.)

THEN f IS 1-1 SINCE $f(a) = f(b) \Rightarrow -a = -b \Rightarrow a = b$, AND

f IS ONTO SINCE $y \in (-1,\infty) \Rightarrow y > -1 \Rightarrow -y < 1$ WITH $f(-y) = y$.

(5) a) LET $f: A \to B$ BE BIJECTIVE AND LET $g: B \to A$.

THEN $g = f^{-1}$ IFF $g \circ f = I_A$ OR $f \circ g = I_B$.

PF $\Rightarrow$ IF $g = f^{-1}$, THEN $g \circ f = I_A$ AND $f \circ g = I_B$ BY TH. 4.2.4.

$\Leftarrow$ SINCE f IS BIJECTIVE, $Dom(g) = B = Rng(f) = Dom(f^{-1})$.

1) IF $g \circ f = I_A$, THEN $g = g \circ I_B = g \circ (f \circ f^{-1}) = (g \circ f) \circ f^{-1}$
$= I_A \circ f^{-1} = f^{-1}$.

2) IF $f \circ g = I_B$, THEN $g = I_A \circ g = (f^{-1} \circ f) \circ g = f^{-1} \circ (f \circ g)$
$= f^{-1} \circ I_B = f^{-1}$.

THEREFORE IF $g \circ f = I_A$ OR $f \circ g = I_B$,

THEN $g = f^{-1}$.

MATH 108 SOLUTIONS – SEC. 4.4 AND THE GOLD SHEET

(5) b) LET $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, AND LET
$f: A \to B$ WITH $f(1) = 2$, $f(2) = 1$, $f(3) = 3$ AND
$g: B \to A$ WITH $g(1) = 2$, $g(2) = 1$, $g(3) = 3$, AND $g(4) = 3$.
THEN $g \circ f = I_A$, BUT $g \neq f^{-1}$ SINCE g IS NOT 1-1.
(NOTICE THAT $\mathrm{Dom}(g) = B \neq \mathrm{Rng}(f)$.)

(7) LET $g: (0, \infty) \to \mathbb{R}$ WITH $g(x) = \ln x$
AND $f: (2, \infty) \to (0, \infty)$ WITH $f(x) = x - 2$.
SINCE f AND g ARE GIVEN TO BE BIJECTIONS,

b) $\underline{g \circ f}: (2, \infty) \to \mathbb{R}$ IS A BIJECTION WHERE $\underline{(g \circ f)(x)} = g(f(x)) = \underline{\ln(x-2)}$.

c) $\underline{(g \circ f)^{-1} = f^{-1} \circ g^{-1}}: \mathbb{R} \to (2, \infty)$ IS A BIJECTION WHERE
$\underline{(f^{-1} \circ g^{-1})(x)} = f^{-1}(g^{-1}(x)) = f^{-1}(e^x) = \underline{e^x + 2}$.

d) $\underline{g^{-1}}: \mathbb{R} \to (0, \infty)$ IS A BIJECTION WHERE $\underline{g^{-1}(x) = e^x}$.

GOLD – (3) $f: (-1, 1) \to \mathbb{R}$ WITH $f(x) = \dfrac{x}{1 - x^2}$ IS BIJECTIVE.

PF i) SINCE $f'(x) = \dfrac{(1-x^2) \cdot 1 - x(-2x)}{(1-x^2)^2} = \dfrac{1 + x^2}{(1-x^2)^2} > 0$ FOR ALL x IN $(-1, 1)$,
f IS INCREASING ON $(-1, 1)$ AND THEREFORE IS $\underline{1-1}$.

OR LET $\underline{f(x) = f(y)}$ WITH $x, y \in (-1, 1)$.
THEN $\dfrac{x}{1-x^2} = \dfrac{y}{1-y^2} \Rightarrow x(1-y^2) = y(1-x^2) \Rightarrow x - xy^2 = y - yx^2 \Rightarrow$
$x - y + x^2 y - xy^2 = 0 \Rightarrow (x-y) + xy(x-y) = 0 \Rightarrow (x-y)(1+xy) = 0 \Rightarrow$
$\underline{x = y}$ SINCE $|x| < 1$ AND $|y| < 1 \Rightarrow |xy| < 1 \Rightarrow xy \neq -1 \Rightarrow 1 + xy \neq 0$.

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ii) LET $y \in \mathbb{R}$.
SINCE $\lim_{x \to 1^-} f(x) = \infty$, THERE IS A NUMBER $b \in (-1, 1)$ WITH $\underline{f(b) > y}$.
SINCE $\lim_{x \to -1^+} f(x) = -\infty$, THERE IS AN $a \in (-1, 1)$ WITH $a < b$ AND $\underline{f(a) < y}$.
SINCE f IS CONTINUOUS ON $[a, b]$, BY THE INTERMEDIATE VALUE TH.
THERE IS A NUMBER $c \in (a, b)$ WITH $\underline{f(c) = y}$.
THEREFORE $y \in \mathrm{Rng}(f)$, SO $\mathrm{Rng}(f) = \mathbb{R}$ AND f IS $\underline{\text{ONTO}}$.