MATH 108 SOLUTIONS – SEC. 5.1

① THE RELATION $\approx$ IS AN EQUIVALENCE RELATION ON THE SET OF ALL SETS IN A UNIVERSE $\mathcal{U}$.

PF 1) IF $A \in \mathcal{U}$, THEN $A \approx A$ SINCE $i_A : A \to A$ IS BIJECTIVE; SO THE RELATION IS REFLEXIVE.

2) IF $A \approx B$, THEN THERE IS A BIJECTION $f : A \to B$. THEN $f^{-1} : B \to A$ IS BIJECTIVE ALSO, SO $B \approx A$ AND THE RELATION IS SYMMETRIC.

3) IF $A \approx B$ AND $B \approx C$, THEN THERE ARE BIJECTIONS $f : A \to B$ AND $g : B \to C$. THEN $g \circ f : A \to C$ IS BIJECTIVE, SO $A \approx C$ AND THUS THE RELATION IS TRANSITIVE.

④ LET $f(x) = \left(\frac{d-c}{b-a}\right)(x-a)+c$. THEN $f : (a,b) \to (c,d)$ IS BIJECTIVE.

PF 0) NOTICE THAT IF $x \in (a,b)$, THEN $a < x < b$ SO $0 < x-a < b-a \Rightarrow$ $0 < \left(\frac{d-c}{b-a}\right)(x-a) < d-c \Rightarrow c < f(x) < d$, SO $f : (a,b) \to (c,d)$.

1) IF $f(x) = f(y)$, THEN $\left(\frac{d-c}{b-a}\right)(x-a)+c = \left(\frac{d-c}{b-a}\right)(y-a)+c \Rightarrow \left(\frac{d-c}{b-a}\right)(x-a) = \left(\frac{d-c}{b-a}\right)(y-a)$ $\Rightarrow x-a = y-a \Rightarrow x = y$; SO f IS 1-1.

2) LET $y \in (c,d)$, SO $c < y < d$ AND THUS $0 < y-c < d-c$. IF $x = \left(\frac{b-a}{d-c}\right)(y-c)+a$, THEN $0 < \left(\frac{b-a}{d-c}\right)(y-c) < b-a$ SO $a < x < b$ WITH $f(x) = \left(\frac{d-c}{b-a}\right)(x-a)+c = (y-c)+c = y$. THEREFORE $y \in \text{Rng}(f)$, SO f IS ONTO.

⑤ LET $h : A \to C$ AND $g : B \to D$ BE BIJECTIONS. THEN $f : A \times B \to C \times D$ DEFINED BY $f(a,b) = (h(a), g(b))$ IS BIJECTIVE.

PF 1) LET $f(a,b) = f(a',b')$, SO $(h(a),g(b)) = (h(a'),g(b'))$. THEN $h(a) = h(a')$ AND $g(b) = g(b')$, SO $a = a'$ AND $b = b'$ SINCE h AND g ARE 1-1 AND THEREFORE $(a,b) = (a',b')$. THUS f IS 1-1.

2) LET $(c,d) \in C \times D$. SINCE h IS SURJECTIVE, $c = h(a)$ FOR SOME $a \in A$; AND SINCE g IS SURJECTIVE, $d = g(b)$ FOR SOME $b \in B$. THEN $f(a,b) = (h(a),g(b)) = (c,d)$, SO f IS SURJECTIVE.

⑦b IF A IS INFINITE AND $A \subseteq B$, THEN B IS INFINITE.

PF (BY CONTRADICTION)
SUPPOSE INSTEAD THAT B IS FINITE. SINCE EVERY SUBSET OF A FINITE SET IS FINITE, A WOULD BE FINITE; AND THIS GIVES A CONTRADICTION. THEREFORE B IS INFINITE.

⑫ IF A IS FINITE AND B IS INFINITE, THEN $B-A$ IS INFINITE.

PF (BY CONTRADICTION)
SUPPOSE INSTEAD THAT $B-A$ IS FINITE. THEN $B = A \cup (B-A)$ WOULD BE FINITE, WHICH GIVES A CONTRADICTION; SO $B-A$ IS INFINITE.

MATH 108 SOLUTIONS – SEC. 5.1 AND 5.2

5.1 – 20 IF $A \approx \mathbb{N}_n$ AND $A \approx \mathbb{N}_m$, THEN $n = m$.

PF SINCE $A \approx \mathbb{N}_n$ AND $A \approx \mathbb{N}_m$, $\mathbb{N}_n \approx \mathbb{N}_m$ SO THERE IS A BIJECTION $f: \mathbb{N}_n \to \mathbb{N}_m$.
SINCE f IS 1-1, $n \leq m$ BY THE PIGEONHOLE PRINCIPLE; AND
SINCE $f^{-1}: \mathbb{N}_m \to \mathbb{N}_n$ IS 1-1, $m \leq n$ BY THE PIGEONHOLE PRINCIPLE.
THEREFORE $m = n$.

5.2 – 2 c) $(0, .001)$ IS INFINITE.

PF SINCE $(0, .001)$ CONTAINS THE DENUMERABLE SET $\left\{ \frac{1}{(10{,}000)\,n} \right\}$,
$(0, .001)$ IS INFINITE.

ALT PF DEFINE $f: (0, .001) \to (0, .0005)$ BY $f(x) = \frac{x}{2}$.
THEN f IS 1-1 AND ONTO, SO $(0, .001)$ IS EQUIVALENT TO A PROPER SUBSET OF ITSELF AND THEREFORE IS INFINITE.

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3 e) $\{x : x \in \mathbb{Z} \text{ AND } x < -12\}$ IS DENUMERABLE.

PF LET $T = \{x : x \in \mathbb{Z} \text{ AND } x < -12\}$, AND DEFINE $f: \mathbb{N} \to T$ BY
$f(n) = -n - 12$. THEN f IS 1-1 SINCE $f(m) = f(n) \Rightarrow -m - 12 = -n - 12$
$\Rightarrow -m = -n \Rightarrow m = n$, AND f IS ONTO SINCE $x \in T \Rightarrow -x > 12$
$\Rightarrow -x \geq 13 \Rightarrow -x - 12 \geq 1$ WITH $f(-x-12) = -(-x-12) - 12 = x$ SO
$x \in Rng(f)$. THEREFORE $\mathbb{N} \approx T$, SO T IS DENUMERABLE.

f) $\mathbb{N} - \{5, 6\}$ IS DENUMERABLE.

PF LET $T = \mathbb{N} - \{5, 6\}$, AND DEFINE $f: \mathbb{N} \to T$ BY $f(n) = \begin{cases} n, & \text{IF } n \leq 4 \\ n+2, & \text{IF } n \geq 5. \end{cases}$

1) LET $f(m) = f(n)$. IF $f(m) \leq 4$, THEN $m = n$.
IF $f(m) \geq 7$, THEN $m + 2 = n + 2$ SO $m = n$. THEREFORE f IS 1-1.

2) LET $m \in T$. IF $m \leq 4$, THEN $f(m) = m$; AND IF $m \geq 7$, THEN
$m - 2 \geq 5$ WITH $f(m-2) = m$. THEREFORE $m \in Rng(f)$ IN EITHER CASE,
SO f IS ONTO.

SINCE $\mathbb{N} \approx T$, T IS DENUMERABLE.

h) $T = \{x \in \mathbb{Z} : x \equiv 1 \pmod 5\}$ IS DENUMERABLE.

PF ~~IF $x \in T$, THEN $x \equiv 1 \pmod 5$ SO $5 \mid (x-1)$ AND THEREFORE~~
~~$x - 1 = 5k$ FOR SOME $k \in \mathbb{Z}$. SINCE $x = 5k+1$,~~
DEFINE $f: \mathbb{Z} \to T$ BY $f(k) = 5k + 1$, WHERE $5k+1 \in T$ SINCE $5k+1 \equiv 1 \pmod 5$ BECAUSE $5 \mid (5k+1) - 1$.

1) IF $f(k) = f(l)$, THEN $5k + 1 = 5l + 1$ SO $5k = 5l$ AND $k = l$;
SO f IS 1-1.

2) IF $x \in T$, THEN $x \equiv 1 \pmod 5$ SO $5 \mid (x-1)$ AND THEREFORE
$x - 1 = 5k$ FOR SOME $k \in \mathbb{Z}$; SO $f(k) = 5k + 1 = x$ AND
HENCE $x \in Rng(f)$. THEREFORE f IS ONTO.

THEREFORE $\mathbb{Z} \approx T$, SO T IS DENUMERABLE SINCE $\mathbb{Z}$ IS DENUMERABLE.