(4) a) Let $f(x) = \frac{1}{x}$.

1) Then $f: (0,1) \to (1,\infty)$ since $0 < x < 1 \Rightarrow f(x) = \frac{1}{x} > 1$.

2) Since $f(x) = f(y) \Rightarrow \frac{1}{x} = \frac{1}{y} \Rightarrow x = y$, f is 1-1.

3) Let $y \in (1,\infty)$, and let $x = \frac{1}{y}$. Since $y > 1$, $0 < x < 1$ so $x \in (0,1)$ with $f(x) = \frac{1}{x} = y$. Therefore $y \in Rng(f)$, so $Rng(f) = (1,\infty)$ and f is onto.

Since $(0,1) \approx (1,\infty)$, $(1,\infty)$ has cardinality c.

REMARK We could also use $f(x) = 1 - \ln x$ to show that $(0,1) \approx (1,\infty)$.

b) Let $f(x) = \frac{1}{x} + a - 1$.

1) Then $f: (0,1) \longrightarrow (a,\infty)$ since $0 < x < 1 \Rightarrow \frac{1}{x} > 1 \Rightarrow f(x) = \frac{1}{x} + a - 1 > a$.

2) Since $f(x) = f(y) \Rightarrow \frac{1}{x} + a - 1 = \frac{1}{y} + a - 1 \Rightarrow \frac{1}{x} = \frac{1}{y} \Rightarrow x = y$, f is 1-1.

3) Let $y \in (a,\infty)$, and let $x = \frac{1}{y-a+1}$. Since $y > a$, $y - a + 1 > 1$ so $x = \frac{1}{y-a+1} \in (0,1)$ with $f(x) = \frac{1}{x} + a - 1 = (y-a+1) + a - 1 = y$.
Therefore $y \in Rng(f)$, so $Rng(f) = (a,\infty)$ and f is onto.

Since $(0,1) \approx (a,\infty)$, (a,∞) has cardinality c.

REMARK We could also use $f(x) = a - \ln x$ to show that $(0,1) \approx (a,\infty)$

c) Let $f(x) = \ln x + b$

1) Then $f: (0,1) \longrightarrow (-\infty, b)$ since $0 < x < 1 \Rightarrow \ln x < 0 \Rightarrow \ln x + b < b$.

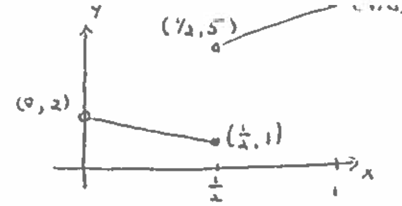
2) If $f(x) = f(y)$, then $\ln x + b = \ln y + b$ so $\ln x = \ln y$ and hence $x = y$; so f is 1-1.

3) Let $y \in (-\infty, b)$, and let $x = e^{y-b}$.
Then $x > 0$, and $x < 1$ since $y < b \Rightarrow y - b < 0 \Rightarrow e^{y-b} < 1$.
Since $f(x) = \ln x + b = (y-b) + b = y$, $y \in Rng(f)$;
so $Rng(f) = (-\infty, b)$ and f is onto.

Since $(0,1) \approx (-\infty, b)$, $(-\infty, b)$ has cardinality c.

REMARK We could also use $f(x) = b + 1 - \frac{1}{x}$ to show that $(0,1) \longrightarrow (-\infty, b)$

MATH 108 SOLUTIONS – SEC. 5.2



④ d) LET $f(x) = \begin{cases} 2-2x, & \text{IF } 0 < x \le \frac{1}{2} \\ 2x+4, & \text{IF } \frac{1}{2} < x < 1. \end{cases}$

1) a) IF $0 < x \le \frac{1}{2}$, THEN $0 < 2x \le 1$ SO $1 \le 2-2x < 2$
AND THEREFORE $f(x) \in [1,2)$.

b) IF $\frac{1}{2} < x < 1$, THEN $1 < 2x < 2$ SO $5 < 2x+4 < 6$ AND $f(x) \in (5,6)$.

THEREFORE $f : (0,1) \longrightarrow [1,2) \cup (5,6)$.

2) LET $f(x) = f(y)$.

a) IF $x, y \in (0, \frac{1}{2}]$, THEN $2-2x = 2-2y \Rightarrow -2x = -2y \Rightarrow x = y$.

b) IF $x, y \in (\frac{1}{2}, 1)$, THEN $2x+4 = 2y+4 \Rightarrow 2x = 2y \Rightarrow x = y$.

c) IF $x \le \frac{1}{2}$ AND $y > \frac{1}{2}$, THEN $f(x) < 2$ AND $f(y) > 5$, WHICH GIVES A CONTRADICTION;
AND SIMILARLY $y \le \frac{1}{2}$ AND $x > \frac{1}{2}$ GIVES A CONTRADICTION.

THEREFORE f IS 1-1 SINCE $f(x) = f(y) \Rightarrow x = y$.

3) LET $y \in [1,2) \cup (5,6)$.

a) IF $1 \le y < 2$, LET $x = \frac{1}{2}(2-y)$. SINCE $1 \le y < 2$, $0 < 2-y \le 1$ SO $0 < x \le \frac{1}{2}$
WITH $f(x) = 2-2x = 2-(2-y) = y$ AND THEREFORE $y \in \operatorname{Rng}(f)$.

b) IF $5 < y < 6$, LET $x = \frac{1}{2}(y-4)$. SINCE $5 < y < 6$, $1 < y-4 < 2$ SO $\frac{1}{2} < x < 1$
WITH $f(x) = 2x+4 = (y-4)+4 = y$ AND THEREFORE $y \in \operatorname{Rng}(f)$,

THUS $\operatorname{Rng}(f) = [1,2) \cup (5,6)$, SO f IS ONTO.

SINCE $(0,1) \approx [1,2) \cup (5,6)$, $[1,2) \cup (5,6)$ HAS CARDINALITY c.

REMARK WE COULD ALSO USE $f(x) = \begin{cases} 2x+5, & \text{IF } 0 < x < \frac{1}{2} \\ 2x, & \text{IF } \frac{1}{2} \le x < 1 \end{cases}$ TO SHOW THIS.

⑤ a) IF A IS COUNTABLE, THEN A IS INFINITE. FALSE

b) IF A IS DENUMERABLE, THEN A IS COUNTABLE. TRUE

c) IF A IS FINITE, THEN A IS DENUMERABLE. FALSE

d) IF A IS UNCOUNTABLE, THEN A IS NOT DENUMERABLE. TRUE
REMARK THIS IS THE CONTRAPOSITIVE OF b).

e) IF A IS UNCOUNTABLE, THEN A IS NOT FINITE. TRUE

f) IF A IS NOT DENUMERABLE, THEN A IS UNCOUNTABLE. FALSE

⑧ a) LET $A = \{2n : n \in \mathbb{N}\} = \{2, 4, 6, 8, \dots\}$
AND $B = \{3n : n \in \mathbb{N}\} = \{3, 6, 9, 12, \dots\}$.
THEN $A \cap B = \{6n : n \in \mathbb{N}\} = \{6, 12, 18, 24, \dots\}$ IS DENUMERABLE.

b) LET $A = \{2n-1 : n \in \mathbb{N}\} = \{1, 3, 5, 7, \dots\}$
AND $B = \{2^{n-1} : n \in \mathbb{N}\} = \{1, 2, 4, 8, \dots\}$
THEN $A \cap B = \{1\}$ IS FINITE.

SOLUTIONS – SEC. 5.2 AND 5.3

5.2 – 8 c) LET $A = \{2n : n \in \mathbb{N}\} = \{2, 4, 6, 8, 10, \ldots\}$

AND $B = \{3n : n \in \mathbb{N}\} = \{3, 6, 9, 12, 15, \ldots\}$.

THEN $A - B = \{2, 4, 8, 10, 14, 16, 20, \ldots\}$ IS DENUMERABLE.

10 SHOW THAT $f : (0,1) \to \mathbb{R}$ DEFINED BY $f(x) = \dfrac{2x-1}{x(x-1)}$ IS BIJECTIVE.

PF 1) $$f'(x) = \frac{x(x-1)\cdot 2 - (2x-1)(2x-1)}{x^2(x-1)^2} = \frac{2x^2 - 2x - (4x^2 - 4x + 1)}{x^2(x-1)^2} = \frac{-2x^2 + 2x - 1}{x^2(x-1)^2}$$

$$= \frac{-2(x^2 - x + \frac{1}{4}) - \frac{1}{2}}{x^2(x-1)^2} = \frac{-2(x-\frac{1}{2})^2 - \frac{1}{2}}{x^2(x-1)^2} < 0 \text{ FOR } x \in (0,1),$$

SO f IS DECREASING ON $(0,1)$ AND THEREFORE IS 1-1.

2) LET $y \in \mathbb{R}$.

a) SINCE $\lim_{x \to 0^+} f(x) = \infty$, THERE IS AN $a \in (0,1)$ WITH $f(a) > y$.

b) SINCE $\lim_{x \to 1^-} f(x) = -\infty$, THERE IS A b WITH $a < b < 1$ AND $f(b) < y$.

c) SINCE f IS CONTINUOUS ON $[a, b]$, THERE IS A NUMBER c IN (a,b) WITH $f(c) = y$ BY THE INTERMEDIATE VALUE TH, SO $y \in Rng(f)$.

THEREFORE $Rng(f) = \mathbb{R}$, SO f IS ONTO.

5.3 – 4 THE FUNCTION $h : \mathbb{N} \to A \cup B$ WITH $h(n) = \begin{cases} f(\frac{n+1}{2}), & \text{IF } n \text{ IS ODD} \\ g(\frac{n}{2}), & \text{IF } n \text{ IS EVEN} \end{cases}$

IS BIJECTIVE IF $f : \mathbb{N} \to A$ AND $g : \mathbb{N} \to B$ ARE BIJECTIVE.

PF 1) LET $h(m) = h(n)$. SINCE A AND B ARE DISJOINT,

m AND n ARE BOTH ODD OR BOTH EVEN,

a) IF m AND n ARE ODD, THEN

$f(\frac{m+1}{2}) = f(\frac{n+1}{2}) \Rightarrow \frac{m+1}{2} = \frac{n+1}{2}$ SINCE f IS 1-1, SO $m+1 = n+1$ AND $m = n$.

b) IF m AND n ARE EVEN, THEN

$g(\frac{m}{2}) = g(\frac{n}{2}) \Rightarrow \frac{m}{2} = \frac{n}{2}$ SINCE g IS 1-1, SO $m = n$.

THEREFORE h IS 1-1.

2) LET $y \in A \cup B$.

a) IF $y \in A$, THEN $y = f(m)$ FOR SOME $m \in \mathbb{N}$ SINCE f IS ONTO,

IF $n = 2m - 1$, THEN $n \in \mathbb{N}$ AND n IS ODD (SINCE $n = 2(m-1)+1$)

SO $h(n) = f(\frac{n+1}{2}) = f(m) = y$. THUS $y \in Rng(h)$.

b) IF $y \in B$, THEN $y = g(m)$ FOR SOME $m \in \mathbb{N}$ SINCE g IS ONTO,

IF $n = 2m$, THEN n IS EVEN SO $h(n) = g(\frac{n}{2}) = g(m) = y$

AND THEREFORE $y \in Rng(h)$.

THUS $Rng(h) = A \cup B$, SO h IS ONTO.