(9) b) IF $A \subseteq B$ WITH A COUNTABLE AND B UNCOUNTABLE, THEN $B - A$ IS UNCOUNTABLE.

PF (BY CONTRADICTION)

SUPPOSE INSTEAD THAT $B - A$ IS COUNTABLE,
THEN $B = A \cup (B - A)$ WOULD BE COUNTABLE, WHICH GIVES A CONTRADICTION.
THEREFORE $B - A$ IS UNCOUNTABLE.

(10) b) IF $A \subseteq B$ AND A IS DENUMERABLE, THEN B IS DENUMERABLE.

FALSE: LET $A = \mathbb{N}$ AND $B = \mathbb{R}$; THEN $\mathbb{N} \subseteq \mathbb{R}$ WITH $\mathbb{N}$ DENUMERABLE,
BUT $\mathbb{R}$ IS NOT DENUMERABLE.

d) $Q - Z$ IS DENUMERABLE.

PF $Q - Z$ IS AN INFINITE SUBSET OF Q, AND Q IS DENUMERABLE;
SO $Q - Z$ IS DENUMERABLE.

(14) a) THE SET S OF ALL SEQUENCES OF 0'S AND 1'S IS UNCOUNTABLE.

PF (BY CONTRADICTION)

SUPPOSE INSTEAD THAT S IS COUNTABLE. SINCE S IS INFINITE, S IS DENUMERABLE SO THERE IS A BIJECTION $f: \mathbb{N} \to S$.
LET $f(n) = x_{n1} x_{n2} x_{n3} x_{n4} \cdots$ FOR EACH $n \in \mathbb{N}$,

$$
\begin{aligned}
\text{SO}\quad f(1) &= x_{11}\, x_{12}\, x_{13}\, x_{14} \cdots \\
f(2) &= x_{21}\, x_{22}\, x_{23}\, x_{24} \cdots \\
f(3) &= x_{31}\, x_{32}\, x_{33}\, x_{34} \cdots \\
&\vdots
\end{aligned}
$$

DEFINE $T \in S$ BY $T = T_1 T_2 T_3 \cdots$ WHERE $T_n = \begin{cases} 0, & \text{IF } x_{nn} = 1 \\ 1, & \text{IF } x_{nn} = 0. \end{cases}$

THEN T AND $f(n)$ DIFFER IN THE nTH TERM FOR EVERY $n \in \mathbb{N}$,
SO $T \notin \mathrm{Rng}(f)$ AND THEREFORE f IS NOT SURJECTIVE.
THIS GIVES A CONTRADICTION, SO S IS UNCOUNTABLE.

b) THE SET T_n OF SEQUENCES IN S WITH EXACTLY n 1'S IS DENUMERABLE.

PF DEFINE $f: T_n \to \mathbb{R}$ AS FOLLOWS:

IF $\tau \in T_n$ WITH $\tau = \tau_1 \tau_2 \tau_3 \cdots$, LET $f(\tau) = .\tau_1 \tau_2 \tau_3 \cdots$
THEN $f(\tau) \in Q$ SINCE τ HAS ONLY FINITELY MANY NONZERO TERMS,
AND f IS 1-1 (BY UNIQUENESS OF DECIMAL EXPANSIONS).
THEREFORE $T_n \approx f(T_n)$, AND $f(T_n)$ IS DENUMERABLE SINCE IT IS AN INFINITE SUBSET OF Q; SO T_n IS DENUMERABLE.

REMARK THE SAME ARGUMENT SHOWS THAT $T = \bigcup_{n=1}^{\infty} T_n$ IS DENUMERABLE,
SINCE WE CAN DEFINE $f: T \to Q$ THE SAME WAY.

5.3 – 15b) Let A be denumerable, and let T be the set of all 2-element subsets of A. Then T is denumerable.

PF Since A is denumerable, there is a bijection $f : A \to \mathbb{N}$; so define $g : T \to \mathbb{N} \times \mathbb{N}$ by $g(\{a,b\}) = (f(a), f(b))$ where $f(a) < f(b)$.
If $g(\{a,b\}) = g(\{c,d\})$, then $(f(a), f(b)) = (f(c), f(d))$ so $f(a) = f(c)$ and $f(b) = f(d)$. Since f is 1-1, $a = c$ and $b = d$ so $\{a,b\} = \{c,d\}$ and therefore g is 1-1.
Since $g(T)$ is an infinite subset of $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ is denumerable, $g(T)$ is denumerable; so T is denumerable since $T \approx g(T)$.

OR PF Since A is denumerable, there is a bijection $f : A \to \mathbb{N}$; so define $g : T \to \mathbb{N}$ by $g(\{a,b\}) = 2^{f(a)} 3^{f(b)}$ where $f(a) > f(b)$.
If $g(\{a,b\}) = g(\{c,d\})$, then $2^{f(a)} 3^{f(b)} = 2^{f(c)} 3^{f(d)}$ so $f(a) = f(c)$ and $f(b) = f(d)$, since f is 1-1, $a = c$ and $b = d$ so $\{a,b\} = \{c,d\}$ and therefore g is 1-1.
Since $g(T)$ is an infinite subset of $\mathbb{N}$, $g(T)$ is denumerable; so T is denumerable since $T \approx g(T)$.

5.4 – 2 b) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

PF Since $|A| = |B|$ and $|B| = |C|$, there are bijections $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ is a bijection, so $|A| = |C|$.

c) If $|A| \le |B|$ and $|B| \le |C|$, then $|A| \le |C|$.

PF Since $|A| \le |B|$ and $|B| \le |C|$, there are injections $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ is an injection, so $|A| \le |C|$.

3 a) If $|A| \le |B|$ and $|B| = |C|$, then $|A| \le |C|$.

PF Since $|A| \le |B|$, there is an injection $f : A \to B$; and since $|B| = |C|$, there is a bijection $g : B \to C$. Then $g \circ f : A \to C$ is injective, so $|A| \le |C|$.

10 If $f : A \to \mathbb{N}$ is 1-1, then A is countable.

PF Since f is 1-1, $A \approx f(A)$; and $f(A)$ is countable since it is a subset of the countable set $\mathbb{N}$.
Therefore A is countable.

11 c) If $|A| < |B|$ and $|B| \le |C|$, then $|A| < |C|$.

PF 1) Since $|A| \le |B|$ and $|B| \le |C|$, $|A| \le |C|$.
2) If $|A| = |C|$, then $|B| \le |C|$ and $|C| = |A|$ so $|B| \le |A|$ (by # 3a).
Then $|A| \le |B|$ and $|B| \le |A|$, so $|A| = |B|$ by the CSB Th.
This gives a contradiction, so $|A| \ne |C|$.
Since $|A| \le |C|$ and $|A| \ne |C|$, $|A| < |C|$.

(11) d) If $|A| < |B|$ and $|B| < |C|$, then $|A| < |C|$.

PF 1) Since $|A| \le |B|$ and $|B| \le |C|$, $\underline{|A| \le |C|}$.

2) If $|A| = |C|$, then $|C| = |A|$ and $|A| \le |B| \Rightarrow \underline{|C| \le |B|}$ (by #3b).

Then $|B| \le |C|$ and $|C| \le |B|$, so $\underline{|B| = |C|}$ by the CSB Th.

This gives a contradiction, so $\underline{|A| \ne |C|}$.

Since $|A| \le |C|$ and $|A| \ne |C|$, $|A| < |C|$.

(13) b) $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$

PF Since $(0,1) \approx \mathbb{R}$, $(0,1) \times (0,1) \approx \mathbb{R} \times \mathbb{R}$ (by #5 in 5.1).

As shown in class, $\underline{(0,1) \times (0,1) \approx (0,1)}$; so $\mathbb{R} \times \mathbb{R} \approx (0,1)$

and therefore $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$.

c) If $A \subseteq \mathbb{R}$ and $(a,b) \subseteq A$, then $|A| = c$.

PF Since $A \subseteq \mathbb{R}$, $\underline{|A| \le |\mathbb{R}|}$. Since $(a,b) \subseteq A$, $\underline{|(a,b)| \le |A|}$

where $\underline{|(a,b)| = |\mathbb{R}|}$ since $(a,b) \approx (0,1)$ and $(0,1) \approx \mathbb{R}$.

Therefore $\underline{|\mathbb{R}| \le |A|}$, so $\underline{|A| = |\mathbb{R}| = c}$ by the CSB Th.

(16) $\mathcal{H} = \{f : f$ is a function from $[0,1]$ to $[0,1]\}$

a) There is no bijection from $[0,1]$ to $\mathcal{H}$.

PF 1) $\underline{|[0,1]| < |\mathcal{P}([0,1])|}$ by Cantor's Th.

2) Define $g : \mathcal{P}([0,1]) \to \mathcal{H}$ by $g(A) = \chi_A$ (where $\chi_A(t) = \begin{cases} 1, & \text{if } t \in A \\ 0, & \text{if } t \notin A \end{cases}$).

Then g is 1-1 since $\underline{g(A) = g(B)} \Rightarrow \chi_A = \chi_B \Rightarrow \underline{A = B}$

(since $t \in A$ iff $\chi_A(t) = 1$ iff $\chi_B(t) = 1$ iff $t \in B$),

so $\underline{|\mathcal{P}([0,1])| \le |\mathcal{H}|}$.

Therefore $\underline{|[0,1]| < |\mathcal{H}|}$ (by #11c), so there is no bijection from $[0,1]$ to $\mathcal{H}$.

b) $\mathcal{H}$ is uncountable since it has a subset equivalent to $[0,1]$.

PF For each $c \in [0,1]$, let f_c be the constant function given by

$\underline{f_c(x) = c \text{ for all } x \in [0,1]}$.

If $g : [0,1] \to \mathcal{H}$ is defined by $g(c) = f_c$, then g is 1-1;

so $[0,1] \approx g([0,1]) \subseteq \mathcal{H}$.

Therefore $\mathcal{H}$ is uncountable since it has an uncountable subset.