MATH 108 SOLUTIONS - SEC. 7.1 AND 7.2

7.1 - ① THE DISCRETE METRIC $d(x,y)=1$ IF $x \neq y$, AND $d(x,x)=0$ FOR ALL x, DEFINES A METRIC ON ANY SET X.

PF 1) $d(x,y) \geq 0$ FOR ALL x,y IN X SINCE EITHER $d(x,y)=1$ OR $d(x,y)=0$.

2) $d(x,y)=0$ IFF $x=y$, SINCE $x=y \Rightarrow d(x,y)=0$, AND $(\Leftarrow)$
$x \neq y \Rightarrow d(x,y)=1 \neq 0$. $(\Rightarrow)$

3) $d(x,y)=d(y,x)$ SINCE BOTH SIDES ARE 1 IF $x \neq y$, AND BOTH SIDES ARE 0 IF $x=y$.

4) $d(x,z) \leq d(x,y)+d(y,z)$

PF (BY CONTRADICTION)

IF $d(x,z) > d(x,y)+d(y,z)$, THEN $d(x,z)=1$ AND $d(x,y)=0=d(y,z)$;
SO $x \neq z$ BUT $x=y$ AND $y=z$. THIS GIVES A CONTRADICTION,
SO $d(x,z) \leq d(x,y)+d(y,z)$.

7.2 - ① 1) $\emptyset$ AND X ARE CLOSED.

PF USING PROP. 7.2.6, $\emptyset$ IS CLOSED SINCE $\emptyset^c = X$ IS OPEN, AND X IS CLOSED SINCE $X^c = \emptyset$ IS OPEN.

2) IF E_λ IS CLOSED FOR EACH $\lambda \in I$, THEN $E = \bigcap_{\lambda \in I} E_\lambda$ IS CLOSED.

PF $E^c = \left(\bigcap_{\lambda \in I} E_\lambda\right)^c = \bigcup_{\lambda \in I} E_\lambda{}^c$ IS OPEN BY PROP. 7.2.6 SINCE $E_\lambda{}^c$ IS OPEN FOR EACH $\lambda \in I$,

SO E IS CLOSED.

3) IF E_i IS CLOSED FOR $1 \leq i \leq k$, THEN $\bigcup_{i=1}^{k} E_i$ IS CLOSED.

PF IF $E = \bigcup_{i=1}^{k} E_i$, THEN $E^c = \bigcap_{i=1}^{k} E_i{}^c$ IS OPEN BY PROP. 7.2.6 SINCE $E_i{}^c$ IS OPEN FOR EACH i;

SO E IS CLOSED.

② IF (X,d) IS A METRIC SPACE, $x \in X$, AND $\delta > 0$, THEN $C(x,\delta)$ IS CLOSED.

PF LET $C = C(x,\delta) = \{z \in X : d(x,z) \leq \delta\}$; AND LET $y \in C^c$.

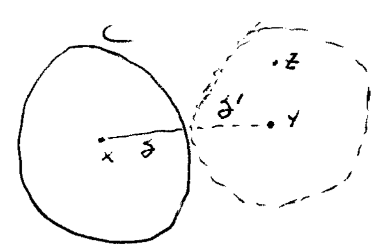
THEN $d(x,y) > \delta$, SO LET $\tau = d(x,y)$ AND LET $\delta' = \tau - \delta$.

IF $z \in B(y,\delta')$, THEN $d(y,z) < \delta'$.

IF $z \in C$, THEN $d(x,z) \leq \delta \Rightarrow d(x,y) \leq d(x,z) + d(z,y) < \delta + \delta' = \tau = d(x,y)$,
WHICH GIVES A CONTRADICTION.

THEREFORE $z \in C^c$, SO $B(y,\delta') \subseteq C^c$.

THUS C^c IS OPEN, SO C IS CLOSED.



MATH 108 SOLUTIONS - SEC. 7.2

(7) a) SHOW THAT E IS CLOSED IFF $\partial E \subseteq E$.

PF $\Rightarrow$ IF E IS CLOSED, THEN $E = \overline{E}$ AND $\partial E = \overline{E} \cap (E^o)^c = E \cap (E^o)^c \subseteq E$.

$\Leftarrow$ SUPPOSE $\partial E \subseteq E$. IF $x \in \overline{E}$, THEN EITHER $x \in E^o \subseteq E$ OR $x \in (E^o)^c$, IN WHICH CASE $x \in \overline{E} \cap (E^o)^c = \partial E \subseteq E$. THEREFORE $\overline{E} \subseteq E$, SO $E = \overline{E}$ AND E IS CLOSED.

b) SHOW THAT U IS OPEN IFF $\partial U \cap U = \emptyset$.

PF $\Rightarrow$ IF U IS OPEN, THEN $U = U^o$ SO $\partial U = \overline{U} \cap (U^o)^c = \overline{U} \cap U^c \subseteq U^c \Rightarrow \partial U \cap U = \emptyset$.

$\Leftarrow$ IF $\partial U \cap U = \emptyset$, THEN $(\overline{U} \cap (U^o)^c) \cap U = \emptyset$ SO $(\overline{U} \cap U) \cap (U^o)^c = \emptyset$ AND THEREFORE $U \cap (U^o)^c = \emptyset$. THEN $U \subseteq U^o$, SO $U = U^o$ AND U IS OPEN. (SEE #8a)

(8) a) SHOW THAT A IS OPEN IFF $A^o = A$.

PF $\Leftarrow$ IF $A^o = A$, THEN A IS OPEN (SINCE A^o IS OPEN).

$\Rightarrow$ IF A IS OPEN, LET $x \in A$. THEN $B(x,\delta) \subseteq A$ FOR SOME $\delta > 0$, SO $x \in A^o$. THEREFORE $A \subseteq A^o$, SO $A = A^o$.

b) SHOW THAT IF U IS AN OPEN SET AND $U \subseteq A$, THEN $U \subseteq A^o$.

PF LET $x \in U$. SINCE U IS OPEN, $B(x,\delta) \subseteq U \subseteq A$ FOR SOME $\delta > 0$, SO $x \in A^o$. THEREFORE $U \subseteq A^o$.

(14) SHOW THAT $A^o = \bigcup \{V : V \text{ IS OPEN AND } V \subseteq A\}$.

PF LET $C = \bigcup \{V : V \text{ IS OPEN AND } V \subseteq A\}$.

1) IF $x \in A^o$, THEN $B(x,\delta) \subseteq A$ FOR SOME $\delta > 0$. SINCE $B(x,\delta)$ IS OPEN, $x \in C$; SO $A^o \subseteq C$.

2) IF $x \in C$, THEN $x \in V$ FOR SOME OPEN SET $V \subseteq A$. SINCE V IS OPEN, $B(x,\delta) \subseteq V \subseteq A$ FOR SOME $\delta > 0$; SO $x \in A^o$. THEREFORE $C \subseteq A^o$.

FROM 1) AND 2), $A^o = C$.

REMARKS

A) IN (7), I AM USING THAT $E \subseteq \overline{E}$.
IN (7)b AND (8)a, I AM USING THAT $A^o \subseteq A$.

B) NOTICE THAT (8)b ALSO FOLLOWS FROM #14.

MATH 108

SOLUTIONS – SEC. 7.3

(1) LET A BE A SUBSET OF A METRIC SPACE (X,d), AND LET
$E = \{x \in X :$ THERE IS A SEQUENCE (x_n) IN A THAT CONVERGES TO $x\}$. SHOW THAT $E = \overline{A}$.

PF a) LET $x \in \overline{A}$. THEN FOR ANY $n \in \mathbb{N}$, $B(x,\frac{1}{n}) \cap A \neq \emptyset$, SO LET $x_n \in B(x,\frac{1}{n}) \cap A$ FOR EACH $n \in \mathbb{N}$.
SINCE $0 \leq d(x_n, x) < \frac{1}{n}$ FOR ALL n, $\lim_{n\to\infty} d(x_n,x) = 0$ BY THE SQUEEZE TH. AND
THEREFORE $\lim_{n\to\infty} x_n = x$. THUS $x \in E$, SO $\underline{\overline{A} \subseteq E}$.

b) LET $x \in E$, AND LET (x_n) BE A SEQUENCE IN A THAT CONVERGES TO x.
THEN FOR ANY $\delta > 0$, THERE IS AN $N \in \mathbb{N}$ SUCH THAT $x_n \in B(x,\delta)$ FOR $n \geq N$.
THEREFORE $B(x,\delta) \cap A \neq \emptyset$ FOR EVERY $\delta > 0$, SO $x \in \overline{A}$. THUS $\underline{E \subseteq \overline{A}}$.

BY PARTS a) AND b), $\underline{E = \overline{A}}$.

(3) A CONVERGENT SEQUENCE IN A METRIC SPACE IS BOUNDED.

PF LET (x_n) BE A SEQUENCE WITH $\lim_{n\to\infty} x_n = x$. THEN THERE IS AN $N \in \mathbb{N}$ SUCH THAT
IF $n \geq N$, THEN $x_n \in B(x,1)$ SO $d(x,x_n) < 1$ (TAKING $\varepsilon = 1$).
IF $r = \max\{d(x,x_1), \ldots, d(x,x_{N-1}), 1\}$, THEN $\underline{d(x,x_n) \leq r}$ FOR ALL $n \in \mathbb{N}$;
SO (x_n) IS BOUNDED (SINCE $E = \{x_n : n \in \mathbb{N}\}$ IS BOUNDED).

(9) LET $E \subseteq X$ BE CLOSED AND LET (x_n) BE A SEQUENCE IN X WITH $\lim_{n\to\infty} x_n = P$.
IF $x_n \in E$ FOR INFINITELY MANY n, SHOW THAT $P \in E$.

PF GIVEN ANY $\delta > 0$, THERE IS AN $N \in \mathbb{N}$ SUCH THAT $x_n \in B(P,\delta)$ FOR $n \geq N$.
THEREFORE $B(P,\delta) \cap E \neq \emptyset$ FOR EVERY $\delta > 0$ (SINCE $x_n \in E$ FOR INFINITELY MANY n),
SO $P \in \overline{E} = E$.

$\boxed{\text{OR}}$ PF (BY CONTRADICTION)
SUPPOSE INSTEAD THAT $P \notin E$. THEN $P \in E^C$ WHERE E^C IS OPEN, SO
$B(P,\delta) \subseteq E^C$ FOR SOME $\delta > 0$. SINCE $\lim_{n\to\infty} x_n = P$, THERE IS AN $N \in \mathbb{N}$
SUCH THAT $x_n \in B(P,\delta) \subseteq E^C$ FOR $n \geq N$. THEREFORE $x_n \notin E$ IF $n \geq N$,
WHICH GIVES A CONTRADICTION. THEREFORE $P \in E$.
