MATH 108 SOLUTIONS – SEC. 1.3 AND 1.4

1.3 – (8)

a) $(\forall x)(x + x \geq x)$ $(\mathbb{R})$ $\underline{F}$: Let $x = -1$, say.

b) $(\forall x)(x + x \geq x)$ $(\mathbb{N})$ $\underline{T}$: If $x \in \mathbb{N}$, $x \geq 0$.

g) $(\forall x)(x^2 + 6x + 5 \geq 0)$ $(\mathbb{R})$ $\underline{F}$: Let $x = -2$, say.

h) $(\forall x)(x^2 + 4x + 5 \geq 0)$ $(\mathbb{R})$ $\underline{T}$: $x^2 + 4x + 5 = (x^2 + 4x + 4) + 1 = (x+2)^2 + 1 \geq 0$

i) $(\forall x)(x^2 - x + 41$ is prime$)$ $(\mathbb{N})$ $\underline{F}$: Let $x = 41$, say.

l) $(\forall x)(\forall y)[x < y \Rightarrow (\exists w)(x < w < y)]$ $(\mathbb{R})$ $\underline{T}$ (Take $w = \frac{x+y}{2}$, say)

(10)

a) $(\forall x)(\exists y)(x + y = 0)$ $\underline{T}$: Take $y = -x$.

b) $(\exists x)(\forall y)(x + y = 0)$ $\underline{F}$: This implies $x = -y$ for every $y \in \mathbb{R}$.

e) $(\forall y)(\exists x)(\forall z)(xy = xz)$ $\underline{T}$: Take $x = 0$.

f) $(\exists x)(\forall y)(x \leq y)$ $\underline{F}$: Take $y = x - 1$, say.

g) $(\forall y)(\exists x)(x \leq y)$ $\underline{T}$: Take $x = y$, say.

1.4 (5)

b) If x is even, then xy is even.

$\underline{\text{PF}}$ Since x is even, $x = 2k$ for some $k \in \mathbb{Z}$.
Then $xy = 2(ky)$ where $ky \in \mathbb{Z}$, so xy is even.

g) If x and y are odd, then xy is odd.

$\underline{\text{PF}}$ Since x and y are odd, $x = 2k + 1$ and $y = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$.
Then $xy = (2k+1)(2\ell+1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1$
where $2k\ell + k + \ell \in \mathbb{Z}$, so xy is odd.

(7)

h) If $a \mid b$, then $a \mid bc$.

$\underline{\text{PF}}$ Since $a \mid b$, $b = ka$ for some integer k.
Then $bc = (ka)c = (kc)a$, so $a \mid bc$ (because $kc \in \mathbb{Z}$).

k) If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

$\underline{\text{PF}}$ Since $a \mid b$ and $c \mid d$, $b = ka$ and $d = \ell c$ for some $k, \ell \in \mathbb{Z}$.
Then $bd = (ka)(\ell c) = (k\ell)(ac)$ where $k\ell \in \mathbb{Z}$, so $ac \mid bd$.

MATH 108 SOLUTIONS – SEC. 1.4 AND 1.5

1.4 – 8 a) n^2+n+3 is odd for all $n \in \mathbb{N}$.

PF 1) If n is even, then $n = 2k$ for some integer k;
so $n^2+n+3 = 4k^2+2k+3 = 2(2k^2+k+1)+1$ is odd since $2k^2+k+1 \in \mathbb{Z}$.

2) If n is odd, then $n = 2k+1$ for some integer k;
so $n^2+n+3 = (2k+1)^2+(2k+1)+3 = 4k^2+6k+5 = 2(2k^2+3k+2)+1$ is odd since $2k^2+3k+2 \in \mathbb{Z}$.

Therefore n^2+n+3 is odd for every $n \in \mathbb{N}$.

9 a) If $x, y \geq 0$, then $\frac{x+y}{2} \geq \sqrt{xy}$.

PF Since $(x-y)^2 \geq 0$, $x^2-2xy+y^2 \geq 0$ and therefore
$x^2+2xy+y^2 \geq 4xy$ (adding $4xy$ to both sides),
then $(x+y)^2 \geq 4xy$, so $x+y \geq 2\sqrt{xy}$ (since $x, y \geq 0$)
and hence $\frac{x+y}{2} \geq \sqrt{xy}$.

REMARK This is the 2-variable case of the ARITHMETIC MEAN – GEOMETRIC MEAN INEQUALITY.

1.5 – 3 c) If x^2 is not divisible by 4, then x is odd.

PF (of the contrapositive)
If x is even, then $x = 2k$ for some integer k;
so $x^2 = 4k^2$ is divisible by 4 (since $k^2 \in \mathbb{Z}$).

10 $\sqrt{5}$ is irrational.

PF (by contradiction)
Suppose instead that $\sqrt{5}$ is rational,
so $\sqrt{5} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ with $n \neq 0$ where m and n have no common factor (greater than 1).
Then $5 = \frac{m^2}{n^2}$, so $m^2 = 5n^2$ and therefore $5 \mid m^2$. Then $5 \mid m$
since 5 is prime, so $m = 5k$ for some $k \in \mathbb{Z}$.
Therefore $m^2 = 25k^2 = 5n^2$, so $5k^2 = n^2$ and thus $5 \mid n^2$.
Then $5 \mid n$ also, and this gives a contradiction
since 5 is a common factor of m and n.
Therefore $\sqrt{5}$ must be irrational.