MATH 108 SOLUTIONS – SEC. 1.6 AND 1.7

1.6 – (2) a) IF $a \mid (b-1)$ AND $a \mid (c-1)$, THEN $a \mid (bc-1)$.

PF SINCE $a \mid (b-1)$ AND $a \mid (c-1)$, $b-1 = ka$ AND $c-1 = \ell a$ FOR SOME $k, \ell \in \mathbb{Z}$,

THEN $b = ka+1$ AND $c = \ell a + 1$, SO

$bc-1 = (ka+1)(\ell a+1) - 1 = k\ell a^2 + ka + \ell a + \cancel{1} - \cancel{1} = a(k\ell a + k + \ell)$

WHERE $k\ell a + k + \ell \in \mathbb{Z}$, SO $a \mid (bc-1)$.

c) IF a IS ODD, $c > 0$, $c \mid a$, AND $c \mid (a+2)$, THEN $c = 1$.

PF SINCE $c \mid a$ AND $c \mid (a+2)$, $c \mid ((a+2)-a)$ SO $c \mid 2$. [SEE EX, P. 34]

THEREFORE $c = 1$ OR $c = 2$, AND $c \neq 2$ SINCE a WOULD BE EVEN IF $2 \mid a$. THUS $c = 1$.

(4) d) $(\forall a,b,c \in \mathbb{Z})(a \mid bc \Rightarrow a \mid b \vee a \mid c)$ F: LET $a = 6$, $b = 2$, $c = 3$

h) $(\forall x > 0)(\exists y > 0)(y < x \wedge (\forall z > 0)(yz \geq z))$ F: LET $x = 1$

(THEN $0 < y < 1 \Rightarrow yz < z \quad \forall z > 0$)

(6) b) $(\exists M \in \mathbb{N})(\forall n \in \mathbb{N})(n > M \Rightarrow \frac{1}{n} < .13)$.

PF LET $M = 8$. IF $n \in \mathbb{N}$ WITH $n > 8$, THEN $\frac{1}{n} < \frac{1}{8} < .13$. (OR USE $M = 7$)

i) $(\exists k \in \mathbb{N})(\forall r \in \mathbb{R})(r > k \Rightarrow \frac{1}{r^2} < .01)$

PF LET $k = 10$. IF $r \in \mathbb{R}$ WITH $r > 10$, THEN $r^2 > 100$ SO $\frac{1}{r^2} < .01$.

1.7 – (3) b) IF $5n+1$ IS EVEN, THEN $2n^2 + 3n + 4$ IS ODD.

PF (OF THE CONTRAPOSITIVE)

IF $2n^2 + 3n + 4$ IS EVEN, THEN $2n^2 + 3n + 4 = 2k$ FOR SOME $k \in \mathbb{Z}$

AND THEREFORE $3n = 2k - 2n^2 - 4 = 2j$ WHERE $j = k - n^2 - 2 \in \mathbb{Z}$,

THEN $5n + 1 = 3n + 2n + 1 = 2j + 2n + 1 = 2(j+n) + 1$ WHERE $j + n \in \mathbb{Z}$,

SO $5n+1$ IS ODD.

c) THE SUM OF 5 CONSECUTIVE INTEGERS IS ALWAYS DIVISIBLE BY 5.

PF IF n IS THE SMALLEST INTEGER, THE SUM IS GIVEN BY

$S = n + (n+1) + (n+2) + (n+3) + (n+4) = 5n + 10 = 5(n+2)$, SO $5 \mid S$.

(5) a) IF $x + y$ IS IRRATIONAL, THEN EITHER x OR y IS IRRATIONAL.

PF (OF THE CONTRAPOSITIVE)

IF x AND y ARE BOTH RATIONAL, THEN $x + y$ IS RATIONAL.

d) FOR EVERY $z \in \mathbb{Q}$, THERE ARE $x, y \notin \mathbb{Q}$ SUCH THAT $x + y = z$.

PF LET $z \in \mathbb{Q}$, AND LET x BE ANY IRRATIONAL,

THEN $z = x + y$ WHERE $y = z - x = z + (-x)$ IS IRRATIONAL BY (5b).

(11) d) A f) A

e) C (ASSUMING THE RIGHT SIDE OF THE EQUATION SHOULD BE $y^2 - 3y$, HALF OF THE PROOF HAS BEEN GIVEN.)

MATH 108 SOLUTIONS – SEC. 2.1

(4) b) T f) F

(7) IF $A \subseteq B$ AND $x \notin B$, THEN $x \notin A$.

PF (BY CONTRADICTION)

ASSUME THAT $A \subseteq B$ AND $x \notin B$, AND THAT $x \in A$.

SINCE $x \in A$ AND $A \subseteq B$, $x \in B$; AND THIS GIVES A CONTRADICTION.

THEREFORE IF $A \subseteq B$ AND $x \notin B$, THEN $x \notin A$.

(8) IF $A \subseteq B$ AND $B \subseteq C$, THEN $A \subseteq C$.

PF IF $x \in A$, THEN $x \in B$ SINCE $A \subseteq B$; AND SINCE $x \in B$ AND $B \subseteq C$, $x \in C$.

THEREFORE $A \subseteq C$.

(15) b) IF $x \in A$, THEN $\{x\} \in \mathcal{P}(A)$. T

c) IF $x \in A$, THEN $\{x\} \subseteq \mathcal{P}(A)$. F

f) IF $B \subseteq A$, THEN $B \in \mathcal{P}(A)$. T

g) IF $B \in \mathcal{P}(A)$, THEN $B \subseteq A$. T

h) IF $C \subseteq B$ AND $B \in \mathcal{P}(A)$, THEN $C \in \mathcal{P}(A)$. T

(SINCE $C \subseteq B$ AND $B \subseteq A$, SO $C \subseteq A$)