MATH 108 SOLUTIONS - SEC. 3.2 AND 3.3

3.2 - 6 a) $x\,S\,y$ IFF $x-y\in\mathbb{Q}$; ON $\mathbb{R}$

1) REFLEXIVE: IF $x\in\mathbb{R}$, THEN $x\,S\,x$ SINCE $x-x=0\in\mathbb{Q}$.

2) SYMMETRIC: IF $x\,S\,y$, THEN $x-y\in\mathbb{Q}$; SO $y-x=-(x-y)\in\mathbb{Q}$ AND $y\,S\,x$.

3) TRANSITIVE: IF $x\,S\,y$ AND $y\,S\,z$, THEN $x-y\in\mathbb{Q}$ AND $y-z\in\mathbb{Q}$; SO $x-z=(x-y)+(y-z)\in\mathbb{Q}$ AND THUS $x\,S\,z$.

THEREFORE S IS AN EQUIVALENCE RELATION ON $\mathbb{R}$.

$[0]=\mathbb{Q}$, $[1]=\mathbb{Q}$, AND $[\sqrt{2}]=\{y+\sqrt{2} : y\in\mathbb{Q}\}$

6 e) $(x,y)\,T\,(a,b)$ IFF $x^2+y^2=a^2+b^2$; ON $\mathbb{R}\times\mathbb{R}$

1) REFLEXIVE: IF $(x,y)\in\mathbb{R}\times\mathbb{R}$, $(x,y)\,T\,(x,y)$ SINCE $x^2+y^2=x^2+y^2$.

2) SYMMETRIC: IF $(x,y)\,T\,(a,b)$, THEN $x^2+y^2=a^2+b^2$ SO $a^2+b^2=x^2+y^2$ AND $(a,b)\,T\,(x,y)$.

3) TRANSITIVE: IF $(x,y)\,T\,(a,b)$ AND $(a,b)\,T\,(c,d)$, THEN $x^2+y^2=a^2+b^2$ AND $a^2+b^2=c^2+d^2$; SO $x^2+y^2=c^2+d^2$ AND HENCE $(x,y)\,T\,(c,d)$.

THEREFORE T IS AN EQUIVALENCE RELATION ON $\mathbb{R}\times\mathbb{R}$.

$[(1,2)]$ CONSISTS OF ALL POINTS ON THE CIRCLE $x^2+y^2=5$, AND

$[(4,0)]$ CONSISTS OF ALL POINTS ON THE CIRCLE $x^2+y^2=16$.

7 $\frac{p}{q}\,R\,\frac{s}{t}$ IFF $pt=qs$; ON $\mathbb{Q}$

1) REFLEXIVE: $\frac{p}{q}\,R\,\frac{p}{q}$ FOR ANY $\frac{p}{q}\in\mathbb{Q}$ SINCE $pq=qp$.

2) SYMMETRIC: IF $\frac{p}{q}\,R\,\frac{s}{t}$, THEN $pt=qs$ SO $sq=tp$ AND THEREFORE $\frac{s}{t}\,R\,\frac{p}{q}$.

3) TRANSITIVE: IF $\frac{p}{q}\,R\,\frac{s}{t}$ AND $\frac{s}{t}\,R\,\frac{u}{v}$, THEN $pt=qs$ AND $sv=tu$.

THEN $(pt)(sv)=(qs)(tu)$, SO $pv=qu$ IF $s\neq 0$; AND $s=0$ IMPLIES

$pt=0=tu \Rightarrow p=0=u \Rightarrow pv=0=qu$.

THEREFORE $pv=qu$ IN EITHER CASE, SO $\frac{p}{q}\,R\,\frac{u}{v}$.

THEREFORE R IS AN EQUIVALENCE RELATION ON $\mathbb{Q}$.

$\left[\frac{2}{3}\right]$ CONSISTS OF ALL FRACTIONS $\frac{p}{q}$ WHOSE REDUCED FORM IS $\frac{2}{3}$.

3.3 - 8 b) $\{A_a : a\in\mathbb{R}\}$ IS A PARTITION OF $\mathbb{R}\times\mathbb{R}$, WHERE $A_a=\{(x,y)\in\mathbb{R}^2 : y=a-x^2\}$.

1) IF $a\in\mathbb{R}$, $A_a\neq\emptyset$ SINCE $(0,a)\in A_a$.

2) IF $(x,y)\in\mathbb{R}\times\mathbb{R}$, LET $a=x^2+y$; THEN $y=a-x^2$ SO $(x,y)\in A_a$ AND THEREFORE $\bigcup_{a\in\mathbb{R}} A_a=\mathbb{R}\times\mathbb{R}$. [THIS SHOWS THAT $\mathbb{R}\times\mathbb{R}\subseteq\bigcup_{a\in\mathbb{R}} A_a$, AND $A_a\subseteq\mathbb{R}\times\mathbb{R}$ FOR EACH $a \Rightarrow \bigcup_{a\in\mathbb{R}} A_a\subseteq\mathbb{R}\times\mathbb{R}$.]

3) LET $(x,y)\in A_a\cap A_b$.

THEN $y=a-x^2$ AND $y=b-x^2$, SO $a=x^2+y=b$ AND THUS $A_a=A_b$.

THEREFORE IF $A_a\neq A_b$, THEN $A_a\cap A_b=\emptyset$.

THUS $\{A_a : a\in\mathbb{R}\}$ IS A PARTITION OF $\mathbb{R}\times\mathbb{R}$.

c) $(x,y)\,R\,(u,v)$ IFF $x^2+y=u^2+v$.

(SINCE $(x,y)\in A_a$ AND $(u,v)\in A_a \Rightarrow y=a-x^2$ AND $v=a-u^2$

$\Rightarrow x^2+y=a=u^2+v$)

(9) d) $\mathcal{P} = \{\{1,5\}, \{2,4\}, \{3\}\}$

$R = \{(1,1), (1,5), (5,1), (5,5), (2,4), (4,2), (2,2), (4,4), (3,3)\}$

(10) $x\,Q\,y$ IFF THERE EXISTS $C \in \mathcal{P}$ SUCH THAT $x \in C$ AND $y \in C$.

a) Q IS SYMMETRIC:

IF $x\,Q\,y$, THEN $x \in C$ AND $y \in C$ FOR SOME $C \in \mathcal{P}$;

SO $y \in C$ AND $x \in C$ AND THUS $y\,Q\,x$.

b) Q IS REFLEXIVE:

IF $x \in A$, THEN $x \in C$ FOR SOME $C \in \mathcal{P}$ SINCE $\bigcup_{C \in \mathcal{P}} C = A$.

SINCE $x \in C$ AND $x \in C$, $x\,Q\,x$.

(12) ASSUME THAT R IS REFLEXIVE AND TRANSITIVE BUT NOT SYMMETRIC ON A,

AND LET $R(x) = \{y \in A : x\,R\,y\}$.

THEN $\mathcal{A} = \{R(x) : x \in A\}$ IS NOT ALWAYS A PARTITION OF A:

EX LET $A = \{1, 2, 3\}$, AND

LET $R = \{(1,1), (2,2), (3,3), (1,2)\}$.

THEN $R(1) = \{1,2\}$, $R(2) = \{2\}$, AND $R(3) = \{3\}$;

SO $\mathcal{A} = \{\{1,2\}, \{2\}, \{3\}\}$ IS NOT A PARTITION OF A

SINCE $R(1) \neq R(2)$ BUT $R(1) \cap R(2) \neq \emptyset$.