MATH 108 SOLUTIONS – SEC. 4.1

1 e) THIS IS NOT A FUNCTION SINCE, FOR EXAMPLE, $(1,1)$ AND $(1,2)$ ARE BOTH IN THE RELATION.

j) THIS IS A FUNCTION, WITH $\mathrm{Dom}(f) = \{a,b,c,d\}$ AND WITH CODOMAIN B GIVEN BY $B = \{1,2,3,4\}$ OR $B = \{2,3,4\}$.

3 b) $\{(x,y) \in \mathbb{R}\times\mathbb{R} : y = x^2+5\}$

$\mathrm{Dom}(f) = \mathbb{R}$, $\mathrm{Rng}(f) = [5,\infty)$ CAN TAKE CODOMAIN $= [5,\infty)$ OR CODOMAIN $= [0,\infty)$, SAY.

9 $A = [1,3)$ a) $\chi_A(1) = 1$ d) $\chi_A(2) - \chi_A(.2) = 1 - 0 = 1$

11 a) $f(x) = \ln(3-x) - \ln x$ AND $g(x) = \ln\frac{3-x}{x}$.

$\mathrm{Dom}(f) = \{x : 3-x>0 \text{ AND } x>0\} = \{x : 0<x<3\} = (0,3)$ AND

$\mathrm{Dom}(g) = \{x : \frac{3-x}{x} > 0\} = (0,3) \longleftarrow$

$-$ $\oplus$ $-$

0 3

IF $x=1$, $\frac{3-x}{x} = 2$

SINCE $\mathrm{Dom}(f) = \mathrm{Dom}(g)$ AND

$g(x) = \ln\frac{3-x}{x} = \ln(3-x) - \ln x = f(x)$ FOR $x \in \mathrm{Dom}(f)$, f AND g ARE EQUAL.

14 b) $f : \mathbb{Z}_4 \to \mathbb{Z}_6$ WITH $f(\bar{x}) = [2x+1]$.

SINCE $\bar{0} = \bar{4}$ IN $\mathbb{Z}_4$ AND $f(\bar{0}) = [1] \neq [9] = f(\bar{4})$ IN $\mathbb{Z}_6$, f IS NOT WELL-DEFINED.

d) $f : \mathbb{Z}_8 \to \mathbb{Z}_5$ WITH $f(\bar{x}) = [x+4]$.

SINCE $\bar{0} = \bar{8}$ IN $\mathbb{Z}_8$ AND $f(\bar{0}) = [4] \neq [12] = f(\bar{8})$ IN $\mathbb{Z}_5$, f IS NOT WELL-DEFINED.

e) $f : \mathbb{Z}_5 \to \mathbb{Z}_6$ WITH $f(\bar{x}) = [x-2]$.

SINCE $\bar{2} = \bar{7}$ IN $\mathbb{Z}_5$ AND $f(\bar{2}) = [0] \neq [5] = f(\bar{7})$ IN $\mathbb{Z}_6$, f IS NOT WELL-DEFINED.

15 a) $f : \mathbb{Z}_4 \to \mathbb{Z}_{12}$ WITH $f(\bar{x}) = [3x]$ IS WELL-DEFINED.

PF LET $\bar{x} = \bar{y}$ IN $\mathbb{Z}_4$, SO $x \equiv y \pmod 4$ AND THEREFORE $4 \mid (x-y)$.
THEN $x-y = 4k$ FOR SOME $k \in \mathbb{Z}$, SO $3x-3y = 12k$ AND THEREFORE
$12 \mid (3x-3y)$. SINCE $3x \equiv 3y \pmod{12}$,
$f(\bar{x}) = [3x] = [3y] = f(\bar{y})$ IN $\mathbb{Z}_{12}$.

d) $f : \mathbb{Z}_{12} \to \mathbb{Z}_4$ WITH $f(\bar{x}) = [2x+1]$ IS WELL-DEFINED.

PF LET $\bar{x} = \bar{y}$ IN $\mathbb{Z}_{12}$, SO $x \equiv y \pmod{12}$ AND THEREFORE $12 \mid (x-y)$,
SO
THEN $x-y = 12k$ FOR SOME $k \in \mathbb{Z}$,
$(2x+1) - (2y+1) = 2(x-y) = 24k = 4(6k)$ WITH $6k \in \mathbb{Z}$.
THEREFORE $2x+1 \equiv 2y+1 \pmod 4$,
SO $f(\bar{x}) = [2x+1] = [2y+1] = f(\bar{y})$ IN $\mathbb{Z}_4$.

17 a) FOR EACH ELEMENT $a \in A$, THERE ARE n CHOICES FOR ITS IMAGE $f(a) \in B$;
SO THERE ARE $\underbrace{n \cdot n \cdots n}_{m \text{ FACTORS}} = n^m$ FUNCTIONS FROM A TO B.

MATH 108 SOLUTIONS - SEC. 4.1 AND 4.2

4.1 - (18) Let $f : A \to B$, and let xTy iff $f(x) = f(y)$ for $x, y \in A$.

a) Then T is an equivalence relation on A.

PF 1) If $a \in A$, then aTa since $f(a) = f(a)$; so T is reflexive.

2) If aTc, then $f(a) = f(c)$ so $f(c) = f(a)$ and thus cTa.
Therefore T is symmetric.

3) If aTc and cTd, then $f(a) = f(c)$ and $f(c) = f(d)$ so $f(a) = f(d)$ and hence aTd. Therefore T is transitive.

b) Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.
Then $[0] = \{0\}$, $[2] = \{2, -2\}$, $[4] = \{4, -4\}$.

4.2 - (2) j) $f(x) = \begin{cases} 2x+3, & \text{if } x < 3 \\ x^2, & \text{if } x \geq 3 \end{cases}$ and $g(x) = \begin{cases} 7-2x, & \text{if } x \leq 2 \\ x+1, & \text{if } x > 2. \end{cases}$

1) $(f \circ g)(x) = f(g(x))$ and

a) if $x \leq 2$, $f(g(x)) = f(7-2x) = (7-2x)^2$ since $7-2x \geq 3$ if $x \leq 2$,

b) if $x > 2$, $f(g(x)) = f(x+1) = (x+1)^2$ since $x+1 \geq 3$ if $x > 2$.

Therefore $\boxed{(f \circ g)(x) = \begin{cases} (7-2x)^2, & \text{if } x \leq 2 \\ (x+1)^2, & \text{if } x > 2 \end{cases}}$.

2) $(g \circ f)(x) = g(f(x))$ and

a) if $x < 3$, $g(f(x)) = g(2x+3) = \begin{cases} 7-2(2x+3) = 1-4x, & \text{if } 2x+3 \leq 2 \text{ so } x \leq -\frac{1}{2} \\ (2x+3)+1 = 2x+4, & \text{if } 2x+3 > 2 \text{ so } x > -\frac{1}{2} \end{cases}$

b) if $x \geq 3$, $g(f(x)) = g(x^2) = x^2 + 1$ since $x^2 > 2$ if $x \geq 3$

Therefore $\boxed{(g \circ f)(x) = \begin{cases} 1-4x, & \text{if } x \leq -\frac{1}{2} \\ 2x+4, & \text{if } -\frac{1}{2} < x < 3 \\ x^2+1, & \text{if } x \geq 3 \end{cases}}$.

(7) Let $f : A \to B$ with $\text{Rng}(F) = C$. If f^{-1} is a function, then $f \circ f^{-1} = I_C$.

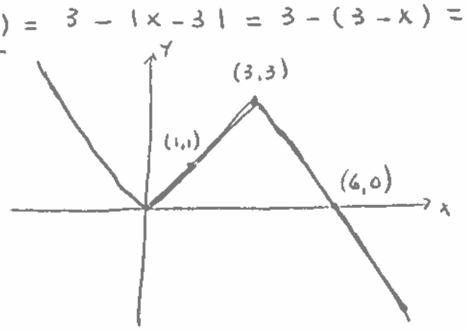
PF 1) $\text{Dom}(f \circ f^{-1}) = \text{Dom}(f^{-1}) = C = \text{Dom}(I_C)$, and

2) $(f \circ f^{-1})(c) = f(f^{-1}(c)) = c = I_C(c)$ for any $c \in C$.

Therefore $f \circ f^{-1} = I_C$.

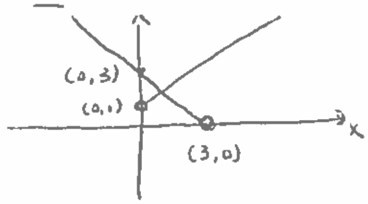
(14) c) $h : (-\infty, 1] \to \mathbb{R}$ with $h(x) = |x|$ and $g : [0, \infty) \to \mathbb{R}$ with $g(x) = 3 - |x-3|$.

On $(-\infty, 1] \cap [0, \infty) = [0, 1]$, $h(x) = |x| = x$ and
$g(x) = 3 - |x-3| = 3 - (3-x) = x$; so $h \cup g$ is a function.



(14) e) $h : (-\infty, 3) \to \mathbb{R}$ with $h(x) = 3 - x$ and $g : (0, \infty) \to \mathbb{R}$ with $g(x) = x + 1$.

$h \cup g$ is NOT a function, since $h(2) = 1 \neq 3 = g(2)$.



(16) d) f is increasing on $(-3, \infty)$, where $f(x) = \dfrac{x-1}{x+3}$.

ALTERNATE PF

Since $f'(x) = \dfrac{(x+3)(1) - (x-1)(1)}{(x+3)^2} = \dfrac{4}{(x+3)^2} > 0$ on $(-3, \infty)$,

f is increasing on $(-3, \infty)$ (as in #19d).

(17) b) Let $f(x) = 1 - x$ and $g(x) = -x$. Then f and g are decreasing on $(-\infty, \infty)$,

but $(f \circ g)(x) = f(g(x)) = f(-x) = 1 + x$ is increasing on $(-\infty, \infty)$;

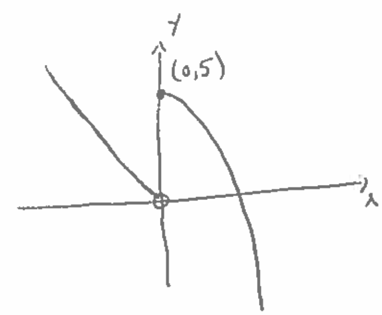
so this gives a counterexample.

e) Let $f(x) = \begin{cases} -x, & \text{if } x < 0 \\ 5 - x^2, & \text{if } x \geq 0. \end{cases}$

Then f is decreasing on $(-\infty, 0)$ and on $[0, \infty)$,

but f is NOT decreasing on $\mathbb{R}$ since $-1 < 0$ but $f(-1) = 1 < 5 = f(0)$;

so this gives a counterexample.



(19) d) A