① A) $p \Rightarrow q \equiv \underline{\sim p \vee q}$

B) $\sim (p \Rightarrow q) \equiv p \underline{\wedge} (\sim q)$

C) $\sim (p \wedge q) \equiv p \underline{\Rightarrow} (\sim q)$

D) $(p \vee q) \wedge (p \vee r) \equiv p \underline{\vee} (q \underline{\wedge} r)$

② PF SINCE $c \mid a$ AND $c \mid b$, $\underline{a = mc}$ AND $\underline{b = nc}$ FOR SOME $m, n \in \mathbb{Z}$.

THEN $\underline{ka + lb} = k(mc) + l(nc) = \underline{(km + ln)c}$ WHERE $km + ln \in \mathbb{Z}$,

SO $c \mid (ka + lb)$.

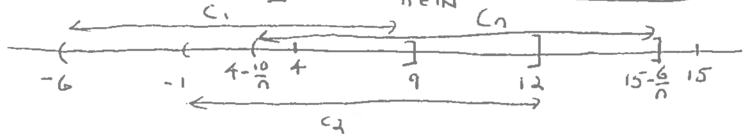
③ PF (OF THE CONTRAPOSITIVE)

IF n IS ODD, THEN $n = 2k+1$ FOR SOME $k \in \mathbb{Z}$;

SO $\underline{(n-6)^2} = (2k-5)^2 = 4k^2 - 20k + 25 = \underline{2(2k^2 - 10k + 12) + 1}$ IS ODD

SINCE $2k^2 - 10k + 12 \in \mathbb{Z}$.

④ $C_n = \left(4 - \frac{10}{n}, 15 - \frac{6}{n}\right]$ A) $\bigcup_{n \in \mathbb{N}} C_n = \underline{(-6, 15)}$ B) $\bigcap_{n \in \mathbb{N}} C_n = \underline{[4, 9]}$



⑤ WE HAVE THAT

$\underline{A-(B-C)} = A \cap (B-C)^c = A \cap (B \cap C^c)^c = A \cap (B^c \cup C) = (A \cap B^c) \cup (A \cap C) = \underline{(A-B) \cup (A \cap C)}$.

$\boxed{OR}$ 1) LET $\underline{x \in A-(B-C)}$, SO $x \in A$ AND $x \notin B - C$. SINCE $x \notin B \cap C^c$, EITHER $x \notin B$ OR $x \in C$. IF $x \notin B$, THEN $x \in A - B$; AND IF $x \in C$, THEN $x \in A \cap C$.

THEREFORE $\underline{x \in (A-B) \cup (A \cap C)}$, SO $\underline{A-(B-C) \subseteq (A-B) \cup (A \cap C)}$.

2) LET $\underline{x \in (A-B) \cup (A \cap C)}$, SO $x \in A - B$ OR $x \in A \cap C$.

a) IF $x \in A - B$, THEN $x \in A$ AND $x \notin B$; SO $x \notin B - C$ SINCE $B - C \subseteq B$.

THEREFORE $x \in A - (B-C)$.

b) IF $x \in A \cap C$, THEN $x \in A$ AND $x \in C$ SO $x \notin B - C$ AND THEREFORE $x \in A - (B-C)$.

THUS $\underline{x \in A-(B-C)}$ IN EITHER CASE, SO $\underline{(A-B) \cup (A \cap C) \subseteq A - (B-C)}$

BY 1) AND 2), $\underline{A-(B-C) = (A-B) \cup (A \cap C)}$.

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⑥ $\sim (\forall \varepsilon)(\varepsilon > 0 \Rightarrow (\exists T)(\forall x)(x > T \Rightarrow |f(x) - L| < \varepsilon)) \equiv$

$\underline{(\exists \varepsilon)(\varepsilon > 0 \wedge (\forall T)(\exists x)(x > T \wedge |f(x) - L| \geq \varepsilon))}$

⑦ A) PF (OF THE CONTRAPOSITIVE)

IF $\sqrt[3]{x}$ IS RATIONAL, THEN $\sqrt[3]{x} = \frac{m}{n}$ WITH $m \in \mathbb{Z}$ AND $n \in \mathbb{N}$;

SO x IS RATIONAL SINCE $x = \frac{m^3}{n^3}$ WITH $m^3 \in \mathbb{Z}$ AND $n^3 \in \mathbb{N}$.

B) THIS IS FALSE: LET $\underline{x = \sqrt{2}}$ AND $\underline{y = -\sqrt{2}}$; THEN x AND y ARE IRRATIONAL,

BUT $x + y = 0$ IS RATIONAL.

REMARK MORE GENERALLY, IF $x \notin \mathbb{Q}$ AND $r \in \mathbb{Q}$, THEN $y = r - x \notin \mathbb{Q}$;

AND $x + y = r \in \mathbb{Q}$.

(8) PROVE THAT $2^{n+1} \ge n^3$ FOR ALL INTEGERS $n \ge 8$ USING THE PMI.

PF 1) THIS IS TRUE FOR $n = 8$, SINCE $2^9 = (2^3)^3 = 8^3$.

2) LET $2^{n+1} \ge n^3$ WHERE $n \in \mathbb{Z}$ WITH $n \ge 8$.

THEN $2^{n+2} = 2(2^{n+1}) \ge 2n^3 = n^3 + n^3 \ge n^3 + 8n^2 = n^3 + 3n^2 + 5n^2$

$\ge n^3 + 3n^2 + 40n = n^3 + 3n^2 + 3n + 37n > n^3 + 3n^2 + 3n + 1 = (n+1)^3$.

THEREFORE $2^{n+1} \ge n^3$ FOR ALL INTEGERS $n \ge 8$ USING THE (GENERALIZED) PMI.

OR USE $2^{n+2} = 2(2^{n+1}) \ge 2n^3$, AND $2n^3 > (n+1)^3$ SINCE

$n \ge 8 \Rightarrow \frac{1}{n} \le \frac{1}{8} \Rightarrow \frac{(n+1)^3}{n^3} = \left(\frac{n+1}{n}\right)^3 = \left(1 + \frac{1}{n}\right)^3 \le \left(1 + \frac{1}{8}\right)^3 = \left(\frac{9}{8}\right)^3 = \frac{729}{512} < 2.$

(9) SINCE $(r-s)^2 + (r-t)^2 + (s-t)^2 \ge 0$,

$r^2 - 2rs + s^2 + r^2 - 2rt + t^2 + s^2 - 2st + t^2 \ge 0$ SO

$2r^2 + 2s^2 + 2t^2 \ge 2rs + 2rt + 2st$ AND THUS $r^2 + s^2 + t^2 \ge rs + rt + st$.

(10) PROVE THAT $\sqrt[3]{5}$ IS IRRATIONAL.

PF (BY CONTRADICTION)

ASSUME INSTEAD THAT $\sqrt[3]{5}$ IS RATIONAL, SO $\sqrt[3]{5} = \frac{m}{n}$ WHERE $m \in \mathbb{Z}$, $n \in \mathbb{N}$,
AND m AND n HAVE NO COMMON FACTOR GREATER THAN 1.

THEN $5 = \frac{m^3}{n^3}$, SO $m^3 = 5n^3$ WHERE $n^3 \in \mathbb{Z}$.

THEREFORE $5 \mid m^3$, SO $5 \mid m$ (SINCE 5 IS PRIME) AND HENCE $m = 5k$ FOR SOME $k \in \mathbb{Z}$.

THEN $m^3 = 125k^3 = 5n^3$, SO $25k^3 = n^3$ AND $5(5k^3) = n^3$ WHERE $5k^3 \in \mathbb{Z}$.

THEREFORE $5 \mid n^3$, SO $5 \mid n$; AND THIS GIVES A CONTRADICTION
SINCE 5 IS A COMMON FACTOR OF m AND n.

THEREFORE $\sqrt[3]{5}$ IS IRRATIONAL.

OR PF (BY CONTRADICTION)

ASSUME INSTEAD THAT $\sqrt[3]{5}$ IS RATIONAL, SO $\sqrt[3]{5} = \frac{m}{n}$ WHERE $m \in \mathbb{Z}$, $n \in \mathbb{N}$,
AND m AND n HAVE NO COMMON PRIME FACTOR.

THEN $5 = \frac{m^3}{n^3}$, SO $m^3 = 5n^3$.

IF $n > 1$, LET p BE A PRIME FACTOR OF n;

THEN $p \mid 5n^3$, SO $p \mid m^3$ AND THEREFORE $p \mid m$ (SINCE p IS PRIME).

THIS GIVES A CONTRADICTION, SO $n = 1$ AND $m^3 = 5$.

SINCE $1^3 < 5 < 2^3$, THIS GIVES A CONTRADICTION;

SO $\sqrt[3]{5}$ IS IRRATIONAL.

(11) Use the WOP to show that $\sqrt{n}$ is irrational for every $n \in \mathbb{N}$ which is not a perfect square (so $\sqrt{n} \notin \mathbb{N}$).

PF (by contradiction)

Assume instead that $\sqrt{n}$ is rational, and let $T = \{ b \in \mathbb{N} : \sqrt{n} = \frac{a}{b} \text{ for some } a \in \mathbb{Z} \}$.

Since $\sqrt{n} \in \mathbb{Q}$, $T \neq \emptyset$; so by the WOP T has a smallest element d, and $\sqrt{n} = \frac{c}{d}$ for some $c \in \mathbb{Z}$.

Therefore $n = \frac{c^2}{d^2}$, so $c^2 = nd^2$.

Since $\sqrt{n} \notin \mathbb{Z}$, $\ell < \sqrt{n} < \ell + 1$ for some $\ell \in \mathbb{Z}$;

so $\ell < \frac{c}{d} < \ell + 1 \implies \ell d < c < \ell d + d \implies 0 < c - \ell d < d$.

Since $c(c - \ell d) = c^2 - c\ell d = nd^2 - c\ell d = (nd - c\ell)d$,

$\sqrt{n} = \frac{c}{d} = \frac{nd - c\ell}{c - \ell d}$ so $c - \ell d \in T$ with $c - \ell d < d$,

This gives a contradiction, so $\sqrt{n}$ is irrational when $n \in \mathbb{N}$ is not a perfect square.