MATH 108 ANSWERS – TEST 2

① LET $R = \{(c,c), (d,d), (e,e), (c,d), (d,e), (d,c), (e,d)\}$. (OTHER ANSWERS ARE POSSIBLE.)
SINCE (c,d) AND (d,e) ARE IN R AND $(c,e) \notin R$, R IS NOT TRANSITIVE.

② $f: [0,\infty) \to (5,8]$ WITH $f(x) = \dfrac{5x+8}{x+1}$.

A) f IS 1-1.

PF LET $f(x) = f(y)$, SO $\dfrac{5x+8}{x+1} = \dfrac{5y+8}{y+1} \Rightarrow (5x+8)(y+1) = (5y+8)(x+1) \Rightarrow$

$5xy + 8y + 5x + 8 = 5xy + 8x + 5y + 8 \Rightarrow 3y = 3x \Rightarrow x = y$. THEREFORE f IS 1-1.

OR PF $f'(x) = \dfrac{(x+1)(5) - (5x+8)(1)}{(x+1)^2} = \dfrac{-3}{(x+1)^2} < 0$ ON $[0,\infty)$,
SO f IS DECREASING ON $[0,\infty)$ AND THEREFORE IS 1-1.

B) f IS ONTO.

PF LET $y \in (5,8]$, SO $5 < y \le 8$; AND LET $x = \dfrac{8-y}{y-5}$.

THEN $x \in [0,\infty)$ SINCE $8-y \ge 0$ AND $y-5 > 0$, AND

$$f(x) = \frac{5x+8}{x+1} = \frac{5\left(\frac{8-y}{y-5}\right)+8}{\frac{8-y}{y-5}+1} = \frac{5(8-y)+8(y-5)}{8-y+y-5} = \frac{3y}{3} = y.$$

THEREFORE $y \in Rng(f)$, SO $Rng(f) = (5,8]$.

③ $m R n$ IFF $5 \mid (9m-4n)$, ON $\mathbb{Z}$

A) R IS SYMMETRIC.

PF LET mRn, SO $5 \mid (9m-4n)$ AND THEREFORE $9m-4n = 5k$ FOR SOME $k \in \mathbb{Z}$.
THEN $9n-4m = (5m+5n)-(9m-4n) = 5m+5n-5k = 5(m+n-k)$ WHERE $m+n-k \in \mathbb{Z}$,
SO $5 \mid (9n-4m)$ AND HENCE nRm.

B) R IS TRANSITIVE.

PF LET mRn AND nRp, SO $5 \mid (9m-4n)$ AND $5 \mid (9n-4p)$.
THEN $9m-4n = 5k$ AND $9n-4p = 5\ell$ FOR SOME $k,\ell \in \mathbb{Z}$, SO
$9m-4p = (9m-4n)+(9n-4p)-5n = 5k+5\ell-5n = 5(k+\ell-n)$ WHERE $k+\ell-n \in \mathbb{Z}$,
THUS $5 \mid (9m-4p)$, SO mRp.

④ IF $f: A \to B$ AND $g: B \to C$ ARE 1-1, THEN $g \circ f$ IS 1-1.

PF ASSUME THAT f AND g ARE 1-1, AND LET $(g\circ f)(x) = (g \circ f)(y)$.
THEN $g(f(x)) = g(f(y))$, SO $f(x) = f(y)$ SINCE g IS 1-1 AND THEREFORE
$x = y$ SINCE f IS 1-1. THEREFORE $g \circ f$ IS 1-1.

⑤ A) $f: \mathbb{N} \xrightarrow[\text{onto}]{1-1} \mathbb{Z}$

LET $\boxed{f(n) = \begin{cases} n/2, & \text{IF } n \text{ IS EVEN} \\ (1-n)/2, & \text{IF } n \text{ IS ODD} \end{cases}}$

(OTHER ANSWERS POSSIBLE)

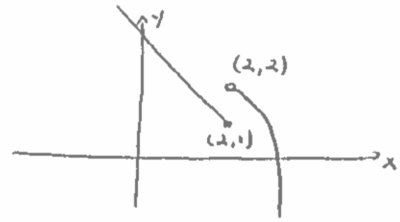
B) $f: \mathbb{N}\times\mathbb{N} \xrightarrow[\text{onto}]{1-1} \mathbb{N}$

LET $\boxed{f(m,n) = 2^{m-1}(2n-1)}$ OR

LET $\boxed{f(m,n) = \dfrac{(m+n-1)(m+n-2)}{2} + m}$.

(OTHER ANSWERS POSSIBLE)

MATH 108 ANSWERS – TEST 2



(6) $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = \begin{cases} 5-2x, & \text{if } x \le 2 \\ 6-x^2, & \text{if } x > 2 \end{cases}$

A) $f(2) = 1 = f(\sqrt{5})$, so f is not 1-1.

B) Let $y \in \mathbb{R}$.

1) If $y \ge 1$, let $x = \frac{1}{2}(5-y)$. Then $y \ge 1 \Rightarrow 5-y \le 4 \Rightarrow x = \frac{1}{2}(5-y) \le 2$,

so $f(x) = 5-2x = 5-(5-y) = y$ and therefore $y \in Rng(f)$.

2) If $y < 1$, let $x = \sqrt{6-y}$. Then $y < 1 \Rightarrow 6-y > 5 \Rightarrow x = \sqrt{6-y} > \sqrt{5} > 2$,

so $f(x) = 6-x^2 = 6-(6-y) = y$ and therefore $y \in Rng(f)$.

Thus $Rng(f) = \mathbb{R}$, so f is onto.

(7) f from $\mathbb{Z}_6$ to $\mathbb{Z}_8$ with $f(\bar{x}) = [5x+3]$

Since $\bar{0} = \bar{6}$ in $\mathbb{Z}_6$ but $f(\bar{0}) = [3] \ne [1] = f(\bar{6})$ in $\mathbb{Z}_8$ since $33 \equiv 1 \pmod 8$ and $3 \not\equiv 1 \pmod 8$,

f is not a well-defined function.

(8) R is an equivalence relation on A, if $c, d \in A$ and cRd, then $[c] = [d]$.

PF 1) Let $a \in [c]$, so cRa. Since cRd, dRc by symmetry and so

dRa by transitivity. Therefore $a \in [d]$, so $[c] \subseteq [d]$.

2) Let $a \in [d]$, so dRa. Since cRd, cRa by transitivity so $a \in [c]$.

Thus $[d] \subseteq [c]$.

Therefore $[c] = [d]$.

(9) Let $f: A \to B$ be 1-1, and let $C \subseteq A$. Then $f(A-C) = f(A) - f(C)$.

PF 1) Let $b \in f(A-C)$, so $b = f(a)$ for some $a \in A-C$.

Then $a \in A$ so $b \in f(A)$, and $a \notin C$.

If $b \in f(C)$, then $b = f(x)$ for some $x \in C$; so $f(a) = b = f(x)$

implies $a = x$ since f is 1-1, and this is impossible since $a \notin C$ and $x \in C$.

Therefore $b \notin f(C)$, so $b \in f(A) - f(C)$. Thus $f(A-C) \subseteq f(A) - f(C)$.

2) Let $b \in f(A) - f(C)$, so $b \in f(A)$ and $b \notin f(C)$.

Then $b = f(a)$ for some $a \in A$, and $a \notin C$ since $b \notin f(C)$.

Therefore $a \in A - C$, so $b \in f(A-C)$. Thus $f(A) - f(C) \subseteq f(A-C)$.

Consequently $f(A-C) = f(A) - f(C)$.

MATH 108 ANSWERS – TEST 2

(10) $G_d = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = d\}$

1) a) IF $d \geq 0$, THEN $(\sqrt{d}, 0) \in G_d$

b) IF $d < 0$, THEN $(0, \sqrt{-d}) \in G_d$.

THEREFORE $\underline{G_d \neq \emptyset}$ FOR ALL $d \in \mathbb{R}$.

2) IF $(x,y) \in \mathbb{R}^2$, LET $d = x^2 - y^2$. THEN $(x,y) \in G_d$, SO $\underline{\bigcup_{d \in \mathbb{R}} G_d = \mathbb{R}^2}$.

3) SUPPOSE THAT $\underline{G_c \cap G_d \neq \emptyset}$ FOR SOME $c, d \in \mathbb{R}$,

AND LET $(x,y) \in G_c \cap G_d$. THEN $c = x^2 - y^2 = d$, SO $\underline{G_c = G_d}$.

THEREFORE $\{G_d : d \in \mathbb{R}\}$ IS A PARTITION OF $\mathbb{R}^2$.

(11) IF $n, d \in \mathbb{N}$, THERE ARE INTEGERS q AND r WITH $\underline{n = qd + r}$ AND $\underline{0 \leq r < d}$.

PF LET $T = \{k \in \mathbb{N} : kd > n\}$.

THEN $T \neq \emptyset$ SINCE $(n+1)d \geq n+1 > n$, SO $n+1 \in T$;

SO T HAS A SMALLEST ELEMENT c BY THE WOP,

LET $\underline{q = c-1}$ AND $\underline{r = n - qd}$. THEN $n = qd + r$, AND

1) $\underline{r < d}$ SINCE $cd > n \Rightarrow (q+1)d > n \Rightarrow qd + d > n \Rightarrow d > r$, AND

2) $\underline{r \geq 0}$ SINCE OTHERWISE $r < 0 \Rightarrow n - qd < 0 \Rightarrow qd > n \Rightarrow q \in T$ WITH $q < c$,

WHICH GIVES A CONTRADICTION.

OR PF LET $T = \{k \in \mathbb{Z} : k \geq 0$ AND $k = n - \ell d$ FOR SOME $\ell \in \mathbb{Z}\}$.

THEN $T \neq \emptyset$ SINCE $n = n - 0(d) \in T$, SO T HAS A SMALLEST ELEMENT r

BY THE WOP (FOR $\mathbb{N} \cup \{0\}$), AND $r = n - qd$ FOR SOME $q \in \mathbb{Z}$.

THEN $n = qd + r$, AND $\underline{r \geq 0}$ SINCE $r \in T$.

IF $r \geq d$, THEN $r - d = (n - qd) - d = n - (q+1)d \geq 0$ SO $r - d \in T$ WITH $r - d < r$,

THIS GIVES A CONTRADICTION, SO $\underline{r < d}$.
