MATH 108 SOLUTIONS – GOLD SHEET

① $A = \{1,2,3,4\}$

a) $A \times A$ HAS $4 \cdot 4 = 16$ ELEMENTS, SO IT HAS $\boxed{2^{16}} = \boxed{65{,}536}$ SUBSETS; SO THIS GIVES THE NUMBER OF RELATIONS ON A.

b) IF $f: A \to A$, THERE ARE 4 CHOICES FOR $f(i)$ FOR EACH i; SO THERE ARE $4 \cdot 4 \cdot 4 \cdot 4 = \boxed{4^4} = \boxed{256}$ FUNCTIONS FROM A TO A.

c) IF $f: A \to A$ IS BIJECTIVE, THEN $f(i) \neq f(j)$ FOR $i \neq j$ (SINCE f IS 1-1); SO THERE ARE $4 \cdot 3 \cdot 2 \cdot 1 = \boxed{24}$ BIJECTIONS FROM A TO A.

d) THE NUMBER OF EQUIVALENCE RELATIONS ON A IS THE SAME AS THE NUMBER OF PARTITIONS OF A, WHICH IS $\boxed{15}$:

SIZES OF SUBSETS	NUMBER OF PARTITIONS	
1/1/1/1	1	$\{\{1\},\{2\},\{3\},\{4\}\}$
4	1	$\{\{1,2,3,4\}\}$
2/2	3	$\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, $\{\{1,4\},\{2,3\}\}$
3/1	4	$\{\{1,2,3\},\{4\}\}$, $\{\{1,2,4\},\{3\}\}$, $\{\{1,3,4\},\{2\}\}$, $\{\{2,3,4\},\{1\}\}$
2/1/1	6	$\{\{1,2\},\{3\},\{4\}\}$, $\{\{1,3\},\{2\},\{4\}\}$, $\{\{1,4\},\{2\},\{3\}\}$, $\{\{2,3\},\{1\},\{4\}\}$, $\{\{2,4\},\{1\},\{3\}\}$, $\{\{3,4\},\{1\},\{2\}\}$

② LET p BE A PRIME AND LET $a, b \in \mathbb{N}$. IF $p \mid ab$, THEN $p \mid a$ OR $p \mid b$.

PF A) ASSUME THAT $p \mid ab$ AND $p \nmid a$. WE MUST SHOW THAT $p \mid b$.

B) LET $T = \{n \in \mathbb{N} : p \mid nb\}$. THEN $a \in T$ SINCE $p \mid ab$ AND $p \in T$ SINCE $p \mid pb$. SINCE $T \neq \emptyset$, T HAS A LEAST ELEMENT c BY THE WOP.

C) IF $n \in T$, THEN $n = qc + r$ FOR SOME INTEGERS q AND r WITH $0 \le r < c$ BY THE DIVISION ALGORITHM.

SINCE $n \in T$, $p \mid nb$ SO $nb = kp$ FOR SOME $k \in \mathbb{Z}$.

SINCE $c \in T$, $p \mid cb$ SO $cb = \ell p$ FOR SOME $\ell \in \mathbb{Z}$.

THEN $rb = (n - qc)b = nb - q(cb) = kp - q(\ell p) = (k - q\ell)p$ WHERE $k - q\ell \in \mathbb{Z}$, SO $p \mid rb$.

IF $r \ge 1$, THEN $r \in T$ WITH $r < c$; SO THIS CONTRADICTS THE FACT THAT c IS THE LEAST ELEMENT OF T.

THEREFORE $r = 0$, SO $n = qc$ AND THEREFORE $c \mid n$.

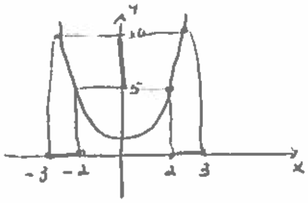
D) BY PART C), $c \mid p$ AND $c \mid a$ SINCE $p \in T$ AND $a \in T$.

SINCE $c \mid p$ AND p IS PRIME, $c = p$ OR $c = 1$.

SINCE $p \nmid a$, $c \neq p$ SO $c = 1$ AND THEREFORE $1 \in T$ IMPLIES $p \mid (1 \cdot b)$ SO $p \mid b$.

SOLUTIONS – SEC. 4.5

(2) e) $f(x) = x^2 + 1$

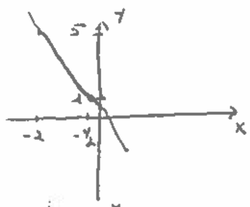


$f^{-1}([5,10]) = \boxed{[-3,-2] \cup [2,3]}$

(OR SOLVE $5 \le x^2 + 1 \le 10$ FOR x)

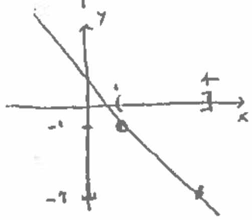
(3) $f(x) = 1 - 2x$

d)



$f^{-1}([2,5]) = \boxed{[-2, -\frac{1}{2}]}$ SINCE $2 \le 1 - 2x \le 5$ IFF $1 \le -2x \le 4$
IFF $-\frac{1}{2} \ge x \ge -2$ IFF $-2 \le x \le -\frac{1}{2}$

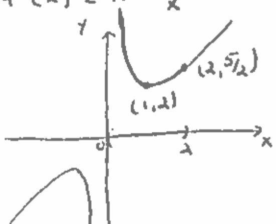
e)



$f((1,4]) = \boxed{[-7,-1)}$

(OR SHOW $1 < x \le 4$ IFF $-7 \le 1 - 2x < -1$)

(4) a) $f(x) = x + \frac{1}{x}$

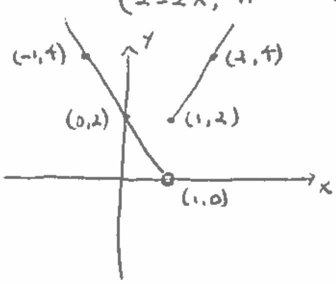


$f((0,2)) = \boxed{[2,\infty)}$

(5) b) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ WITH $f(m,n) = 2^m 3^n$

$f^{-1}(\{5,6,7,8,9,10\}) = \boxed{\{(1,1)\}}$ SINCE ANY NUMBER IN $Rng(f)$ IS A MULTIPLE OF 6.

(7) $f(x) = \begin{cases} 2x, & \text{IF } x \ge 1 \\ 2 - 2x, & \text{IF } x < 1. \end{cases}$



b) LET $A = [-1,0]$ AND $C = [1,2]$.
THEN $A \cap C = \emptyset$ AND $f(A) = [2,4] = f(C)$,
SO $\underline{f(A \cap C) \ne f(A) \cap f(C)}$ (SINCE $\emptyset \ne [2,4]$)

c) LET $D = [1,2]$.
THEN $f(D) = [2,4]$ AND $f^{-1}(f(D)) = [-1,0] \cup [1,2]$,
SO $\underline{D \ne f^{-1}(f(D))}$

d) LET $E = [0,2]$.
THEN $f^{-1}(E) = [0,1]$ AND $f(f^{-1}(E)) = (0,2]$,
SO $\underline{E \ne f(f^{-1}(E))}$

<u>REMARK</u> SEE ALSO (12d) AND (12f).

9 b) LET $f: A \to B$ AND LET C AND D BE SUBSETS OF A. THEN $f(C \cup D) = f(C) \cup f(D)$

PF 1) LET $b \in f(C \cup D)$, SO $b = f(a)$ WHERE $a \in C \cup D$ SO $a \in C$ OR $a \in D$.
IF $a \in C$, THEN $b \in f(C)$; AND IF $a \in D$, THEN $b \in f(D)$.
THEREFORE $b \in f(C) \cup f(D)$, SO $f(C \cup D) \subseteq f(C) \cup f(D)$.

2) LET $b \in f(C) \cup f(D)$, SO $b \in f(C)$ OR $b \in f(D)$. THEN $b = f(c)$ WITH $c \in C$ OR $b = f(d)$ WHERE $d \in D$. SINCE $c \in C \cup D$ AND $d \in C \cup D$, $b \in f(C \cup D)$ IN EITHER CASE; SO $f(C) \cup f(D) \subseteq f(C \cup D)$.

THUS $f(C \cup D) = f(C) \cup f(D)$.

12 LET $f: A \to B$, $D \subseteq A$, $E \subseteq B$.

b) $A - f^{-1}(E) \subseteq f^{-1}(B - E)$

PF LET $a \in A - f^{-1}(E)$. SINCE $a \notin f^{-1}(E)$, $f(a) \notin E$ SO $f(a) \in B - E$.
THEREFORE $a \in f^{-1}(B - E)$, SO $A - f^{-1}(E) \subseteq f^{-1}(B - E)$.

c) $f^{-1}(B - E) \subseteq A - f^{-1}(E)$

PF LET $a \in f^{-1}(B - E)$, SO $f(a) \in B - E$. SINCE $f(a) \notin E$, $a \notin f^{-1}(E)$ SO $a \in A - f^{-1}(E)$. THEREFORE $f^{-1}(B - E) \subseteq A - f^{-1}(E)$.

REMARK $A - f^{-1}(E) = f^{-1}(B - E)$ BY PARTS b) AND c).

e) $D \subseteq f^{-1}(f(D))$

PF LET $d \in D$. THEN $f(d) \in f(D)$, SO $d \in f^{-1}(f(D))$.

f) $D = f^{-1}(f(D))$ IFF $f(A - D) \subseteq B - f(D)$

PF $\Longrightarrow$ (USING THE CONTRAPOSITIVE)
SUPPOSE THAT $f(A - D)$ IS NOT A SUBSET OF $B - f(D)$.
THEN THERE IS AN ELEMENT $f(a) \in f(A - D)$ WHERE $a \in A - D$ SUCH THAT $f(a) \in f(D)$. THEN $a \notin D$ BUT $a \in f^{-1}(f(D))$, SO $D \neq f^{-1}(f(D))$.

$\Longleftarrow$ ASSUME THAT $f(A - D) \subseteq B - f(D)$.
1) $D \subseteq f^{-1}(f(D))$ BY PART e).
2) IF $a \in f^{-1}(f(D))$, THEN $f(a) \in f(D)$.
IF $a \notin D$, THEN $f(a) \in f(A - D) \subseteq B - f(D)$, WHICH GIVES A CONTRADICTION; SO $a \in D$ AND HENCE $f^{-1}(f(D)) \subseteq D$.
THEREFORE $D = f^{-1}(f(D))$.

g) $f(f^{-1}(E)) = E \cap Rng(f)$

PF 1) IF $b \in E \cap Rng(f)$, THEN $b \in E$ AND $b = f(a)$ FOR SOME $a \in A$.
SINCE $f(a) \in E$, $a \in f^{-1}(E)$ AND $b = f(a) \in f(f^{-1}(E))$.
THEREFORE $E \cap Rng(f) \subseteq f(f^{-1}(E))$.

2) IF $b \in f(f^{-1}(E))$, THEN $b = f(a)$ FOR SOME $a \in f^{-1}(E)$;
SO $f(a) \in E \Longrightarrow b \in E$, AND $b = f(a) \Longrightarrow b \in Rng(f)$.
THEREFORE $b \in E \cap Rng(f)$, SO $f(f^{-1}(E)) \subseteq E \cap Rng(f)$.

THUS $f(f^{-1}(E)) = E \cap Rng(f)$.

REMARK NOTICE THAT PART d) FOLLOWS FROM PART g).

(13) Let $f: A \to B$ and let $X, Y \subseteq A$ and $U, V \subseteq B$.

a) $f(X) \subseteq U$ iff $X \subseteq f^{-1}(U)$

PF We have that $f(X) \subseteq U$ iff $f(a) \in U$ for all $a \in X$ iff $a \in f^{-1}(U)$ for all $a \in X$ iff $X \subseteq f^{-1}(U)$.

c) $f^{-1}(U) - f^{-1}(V) = f^{-1}(U - V)$

PF If $a \in A$, then $a \in f^{-1}(U) - f^{-1}(V)$ iff $a \in f^{-1}(U)$ and $a \notin f^{-1}(V)$ iff $f(a) \in U$ and $f(a) \notin V$ iff $f(a) \in U - V$ iff $a \in f^{-1}(U-V)$;
so $f^{-1}(U) - f^{-1}(V) = f^{-1}(U-V)$.

(14) Let $f: A \to B$.

a) If f is 1-1, then $f(X) \cap f(Y) = f(X \cap Y)$ if $X, Y \subseteq A$.

PF 1) $f(X \cap Y) \subseteq f(X) \cap f(Y)$ by part a) of Th. 4.5.1

2) Let $b \in f(X) \cap f(Y)$, so $b = f(c)$ for some $c \in X$ and $b = f(d)$ for some $d \in Y$. Since f is 1-1, $c = d$ so $c \in X \cap Y$. Therefore $b \in f(X \cap Y)$, so $f(X) \cap f(Y) \subseteq f(X \cap Y)$.

Thus $f(X) \cap f(Y) = f(X \cap Y)$ when f is 1-1.

c) If $X \subseteq A$ and f is 1-1, then $f(A - X) = f(A) - f(X)$.

PF 1) Let $b \in f(A) - f(X)$, so $b \in f(A)$ and $b \notin f(X)$. Since $b \in f(A)$, $b = f(a)$ for some $a \in A$; and $a \notin X$ since otherwise b would be in $f(X)$. Then $a \in A - X$, so $b = f(a) \in f(A - X)$. Hence $f(A) - f(X) \subseteq f(A - X)$.

2) Let $b \in f(A - X)$, so $b = f(a)$ where $a \in A - X$ so $a \in A$ and $a \notin X$. If $b \in f(X)$, then $b = f(c)$ for some $c \in X$; and this gives a contradiction since $f(a) = b = f(c) \implies a = c$ since f is 1-1, and $a \notin X$ and $c \in X$.
Therefore $b \in f(A)$ and $b \notin f(X)$, so $b \in f(A) - f(X)$ and consequently $f(A - X) \subseteq f(A) - f(X)$.

Thus $f(A - X) = f(A) - f(X)$ when f is 1-1.