4 b) $3+11+19+\cdots+(8n-5)=4n^2-n$ FOR ALL $n \in \mathbb{N}$.

PF 1) THIS IS TRUE FOR $n=1$, SINCE $3=4(1)^2-1$.

2) ASSUME THAT $3+11+19+\cdots+(8n-5)=4n^2-n$ WHERE $n \in \mathbb{N}$.

THEN $3+11+19+\cdots+(8n-5)+(8n+3)=\left[4n^2-n\right]+8n+3=$

$4n^2+7n+3=4n^2+8n+4-(n+1)=4(n^2+2n+1)-(n+1)=4(n+1)^2-(n+1)$.

THEREFORE $3+11+19+\cdots+(8n-5)=4n^2-n$ FOR ALL $n \in \mathbb{N}$ BY THE PMI.

d) $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$ FOR ALL $n \in \mathbb{N}$.

PF 1) THIS IS TRUE FOR $n=1$, SINCE $1\cdot 1!=1=2!-1$.

2) ASSUME THAT $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$ WHERE $n \in \mathbb{N}$.

THEN $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!+(n+1)(n+1)!=\left[\underline{(n+1)!}-1\right]+\underline{(n+1)(n+1)!}=$

$\underline{(n+2)(n+1)!}-1=(n+2)!-1$.

THEREFORE $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$ FOR ALL $n \in \mathbb{N}$ BY THE PMI.

e) $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$ FOR ALL $n \in \mathbb{N}$.

PF 1) THIS IS TRUE FOR $n=1$, SINCE $1^3=1=\dfrac{1^2\cdot 2^2}{4}$.

2) LET $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$ WHERE $n \in \mathbb{N}$.

THEN $1^3+2^3+3^3+\cdots+n^3+(n+1)^3=\left[\dfrac{n^2(n+1)^2}{4}\right]+(n+1)^3=$

$\dfrac{(n+1)^2}{4}\left[n^2+4(n+1)\right]=\dfrac{(n+1)^2}{4}\left[n^2+4n+4\right]=\dfrac{(n+1)^2(n+2)^2}{4}$.

THEREFORE $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$ FOR ALL $n \in \mathbb{N}$ BY THE PMI.

5 a) n^3+5n+6 IS DIVISIBLE BY 3 FOR ALL $n \in \mathbb{N}$.

PF 1) THIS IS TRUE FOR $n=1$, SINCE $12=1^3+5(1)+6$ IS DIVISIBLE BY 3.

2) ASSUME THAT n^3+5n+6 IS DIVISIBLE BY 3 WHERE $n \in \mathbb{N}$,

SO $n^3+5n+6=3k$ FOR SOME $k \in \mathbb{Z}$.

THEN $(n+1)^3+5(n+1)+6=\underline{n^3}+3n^2+3n+1+\underline{5n}+5+\underline{6}=$

$(n^3+5n+6)+(3n^2+3n+6)=3k+(3n^2+3n+6)=$

$3(k+n^2+n+2)$ WHERE $k+n^2+n+2 \in \mathbb{Z}$, SO $3 \mid \left((n+1)^3+5(n+1)+6\right)$.

THEREFORE n^3+5n+6 IS DIVISIBLE BY 3 FOR ALL $n \in \mathbb{N}$ BY THE PMI.

(5) f) $10^{n+1} + 3\cdot 4^{n-1} + 5$ is divisible by 9 for all $n \in \mathbb{N}$.

PF 1) This is true for $n=1$, since $10^2 + 3 + 5 = 108$ is divisible by 9.

2) Assume that $10^{n+1} + 3\cdot 4^{n-1} + 5$ is divisible by 9 where $n \in \mathbb{N}$,

so $10^{n+1} + 3\cdot 4^{n-1} + 5 = 9k$ for some $k \in \mathbb{Z}$.

Then $10^{n+2} + 3\cdot 4^{n} + 5 = \left[10^{n+1} + 3\cdot 4^{n-1} + 5\right] + \left(10^{n+2} - 10^{n+1}\right) + \left(3\cdot 4^{n} - 3\cdot 4^{n-1}\right) =$

$9k + 9\cdot 10^{n+1} + 9\cdot 4^{n-1} = 9\left(k + 10^{n+1} + 4^{n-1}\right)$ where

$k + 10^{n+1} + 4^{n-1} \in \mathbb{Z}$, so $10^{n+2} + 3\cdot 4^{n} + 5$ is divisible by 9.

Therefore $10^{n+1} + 3\cdot 4^{n-1} + 5$ is divisible by 9 for all $n \in \mathbb{N}$ by the PMI.

i) Let p be a prime, and let $a, n \in \mathbb{N}$. If $p \mid a^n$, then $p \mid a$.

PF (by induction on n)

1) This is clearly true for $n=1$.

2) Assume that this is true for an arbitrary $n \in \mathbb{N}$.

If $p \mid a^{n+1}$, then $p \mid a(a^n)$ so $p \mid a$ or $p \mid a^n$ (since p is prime);

and if $p \mid a^n$, then $p \mid a$ by the induction hypothesis.

Thus if $p \mid a^{n+1}$, then $p \mid a$.

Therefore this statement is true for all $n \in \mathbb{N}$ by the PMI.

m) $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

PF 1) This is true for $n=1$, since $\frac{1}{1^2} = 1 \le 2 - 1$.

2) Assume that $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \le 2 - \frac{1}{n}$ where $n \in \mathbb{N}$.

Then $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \le 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$.

Since $\underline{\frac{1}{(n+1)^2} < \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}}$, $\underline{2 - \frac{1}{n} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n+1}}$,

so $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n+1}$.

Therefore $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$ by the PMI.

(6) a) $n! \ge n^3$ for all integers $n \ge 6$.

PF 1) For $n=6$ this is true, since $6! = 720 > 216 = 6^3$.

2) Let $n! \ge n^3$ where $n \in \mathbb{N}$ with $n \ge 6$.

Then $(n+1)! = (n+1)\,n! \ge (n+1)\,n^3 \ge (n+1)(6n^2) = (n+1)(n^2 + 5n^2) \ge$

$(n+1)(n^2 + 30n) = (n+1)(n^2 + 2n + 28n) > (n+1)(n^2 + 2n + 1)$

$= (n+1)(n+1)^2 = (n+1)^3$.

Therefore $n! \ge n^3$ for all integers $n \ge 6$ by induction.

MATH 108 Sec. 2.4 AND 2.5 – SOLUTIONS

2.4 – 6b) $2^n > n^2$ FOR ALL INTEGERS $n > 4$.

PF 1) THIS IS TRUE FOR $n=5$, SINCE $2^5 = 32 > 25 = 5^2$.

2) LET $2^n > n^2$ WHERE n IS AN INTEGER WITH $n \ge 5$.

THEN $2^{n+1} = 2(2^n) > 2n^2 = n^2 + n^2 \ge n^2 + 5n = n^2 + 2n + 3n > n^2 + 2n + 1 = (n+1)^2$.

THEREFORE $2^n > n^2$ FOR ALL INTEGERS $n > 4$ BY INDUCTION.

2.5 – 1 a) EVERY INTEGER $n \ge 11$ CAN BE WRITTEN AS $n = 2s + 5t$ FOR SOME $s, t \in \mathbb{N}$.

PF 1) THIS IS TRUE FOR $n = 11$, SINCE $11 = 2(3) + 5(1)$,

AND FOR $n = 12$, SINCE $12 = 2(1) + 5(2)$.

2) LET n BE AN INTEGER WITH $n \ge 12$, AND ASSUME THIS STATEMENT IS TRUE FOR ALL INTEGERS k WITH $11 \le k \le n$.

SINCE $n - 1 \ge 11$, $n - 1 = 2s + 5t$ FOR SOME $s, t \in \mathbb{N}$ AND

THEREFORE $n + 1 = 2(s+1) + 5t$.

THEREFORE EVERY INTEGER $n \ge 11$ CAN BE WRITTEN IN THIS FORM BY THE PSI.

3 IF $a_1 = 2$, $a_2 = 4$, AND $a_{n+2} = 5a_{n+1} - 6a_n$ FOR $n \ge 1$, THEN $a_n = 2^n$ FOR ALL $n \in \mathbb{N}$.

PF 1) THIS IS TRUE FOR $n = 1$ AND $n = 2$, SINCE $a_1 = 2^1$ AND $a_2 = 2^2$.

2) LET n BE AN INTEGER WITH $n \ge 2$, AND ASSUME THAT $a_k = 2^k$ FOR ALL INTEGERS k WITH $1 \le k \le n$.

THEN $a_{n+1} = 5a_n - 6a_{n-1} = 5(2^n) - 6(2^{n-1}) = (10-6)2^{n-1} = 4 \cdot 2^{n-1} = 2^{n+1}$.

THEREFORE $a_n = 2^n$ FOR ALL $n \in \mathbb{N}$ BY THE PSI.

4 d) $\sqrt{2}$ IS IRRATIONAL.

PF (BY CONTRADICTION)

ASSUME INSTEAD THAT $\sqrt{2}$ IS RATIONAL. THEN IF $T = \{n \in \mathbb{N} : \sqrt{2} = \frac{m}{n} \text{ FOR SOME } m \in \mathbb{Z}\}$, $T \ne \emptyset$;

SO BY THE WOP, T HAS A SMALLEST ELEMENT b,

AND $\sqrt{2} = \frac{a}{b}$ FOR SOME $a \in \mathbb{Z}$.

THEN $2 = \frac{a^2}{b^2}$, SO $a^2 = 2b^2$.

SINCE $a^2 - ab = 2b^2 - ab = b(2b - a)$,

$$\sqrt{2} = \frac{a}{b} = \frac{2b-a}{a-b} \quad \text{WHERE } a - b \in \mathbb{N} \text{ WITH } a - b < b$$

SINCE $1 < \sqrt{2} < 2 \Rightarrow 1 < \frac{a}{b} < 2 \Rightarrow a > b$ AND $a < 2b$.

THEREFORE $a - b \in T$ WITH $a - b < b$, AND THIS GIVES A CONTRADICTION; SO $\sqrt{2}$ IS IRRATIONAL.

MATH 108 SEC. 2.5 – SOLUTIONS

(13) a) SHOW THAT THE WOP IMPLIES THE PMI.

PF LET S BE A SUBSET OF $\mathbb{N}$ SUCH THAT 1) $1 \in S$ AND

2) FOR ALL $n \in \mathbb{N}$, $n \in S \Rightarrow n+1 \in S$.

WE WILL SHOW USING AN ARGUMENT BY CONTRADICTION THAT $S = \mathbb{N}$:

IF $S \neq \mathbb{N}$, THEN $T = \mathbb{N} - S$ IS NONEMPTY;

SO BY THE WOP T HAS A SMALLEST ELEMENT m.

THEN $m > 1$ SINCE $1 \in S$, SO $m-1 \in \mathbb{N}$ AND $m-1 \notin T$,

THEREFORE $m-1 \in S$, SO $m \in S$; AND THIS GIVES A CONTRADICTION.

THUS $S = \mathbb{N}$.

b) SHOW THAT THE PCI IMPLIES THE WOP.

PF (BY CONTRADICTION)

LET T BE A NONEMPTY SUBSET OF $\mathbb{N}$, AND SUPPOSE THAT T HAS NO SMALLEST ELEMENT. IF $S = \mathbb{N} - T$, THEN 1) $1 \in S$ SINCE OTHERWISE 1 WOULD BE THE SMALLEST ELEMENT OF T.

2) IF $\{1, \ldots, n\} \subseteq S$ WHERE $n \in \mathbb{N}$, THEN $n+1 \in S$ SINCE OTHERWISE $n+1$ WOULD BE THE SMALLEST ELEMENT OF T.

THEREFORE $S = \mathbb{N}$ BY THE PSI, SO $T = \emptyset$; AND THIS GIVES A CONTRADICTION.

THUS EVERY NONEMPTY SUBSET OF $\mathbb{N}$ HAS A SMALLEST ELEMENT.