MATH 108 — ANSWERS – QUIZ 2A

① $\mathcal{P} = \{\{1,2\},\{3,5\},\{4\}\}$

$R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,5),(5,3),(5,5),(4,4)\}$

② LET $R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,c),(b,a),(c,b)\}$, FOR EXAMPLE.

R IS NOT TRANSITIVE SINCE $(a,b) \in R$ AND $(b,c) \in R$ BUT $(a,c) \notin R$.

③ $t \, S \, u$ IFF $t = (2^m u)$ FOR SOME $m \in \mathbb{Z}$, ON $\mathbb{Q}$

1) IF $t \in \mathbb{Q}$, THEN $t \, S \, t$ SINCE $t = 1 \cdot t = 2^0 \cdot t$; SO S IS REFLEXIVE.

2) IF $t \, S \, u$, THEN $t = (2^m) u$ FOR SOME $m \in \mathbb{Z}$;
SO $u = (2^{-m}) t$ WHERE $-m \in \mathbb{Z}$ AND THUS $u \, S \, t$. THEREFORE S IS SYMMETRIC.

3) IF $t \, S \, u$ AND $u \, S \, v$, THEN $t = (2^m) u$ AND $u = (2^n) v$ FOR SOME $m, n \in \mathbb{Z}$.
THEREFORE $t = 2^m (2^n v) = 2^{m+n} v$ WHERE $m+n \in \mathbb{Z}$, SO $t \, S \, v$.
THUS S IS TRANSITIVE.

CONSEQUENTLY S IS AN EQUIVALENCE RELATION ON $\mathbb{Q}$.

④ EVERY INTEGER $n > 1$ CAN BE WRITTEN AS $n = p_1 \cdots p_r$ WHERE $p_1, \ldots, p_r$ ARE PRIMES.

PF 1) THIS IS TRUE FOR $n = 2$, SINCE 2 IS A PRIME.

2) LET $n \in \mathbb{Z}$ WITH $n \geq 2$, AND ASSUME EVERY INTEGER $k \in \{2, \ldots, n\}$ CAN BE WRITTEN IN THIS FORM.

A) IF $n+1$ IS PRIME, THEN IT IS OF THIS FORM.

B) IF $n+1$ IS NOT PRIME, THEN $n+1 = k\ell$ WHERE k AND ℓ ARE INTEGERS WITH $1 < k < n+1$ AND $1 < \ell < n+1$.
BY THE INDUCTION HYPOTHESIS, $k = p_1 \cdots p_r$ AND $\ell = q_1 \cdots q_s$
FOR SOME PRIMES $p_1, \ldots, p_r$ AND $q_1, \ldots, q_s$.
THEREFORE $n+1 = k\ell = p_1 \cdots p_r q_1 \cdots q_s$

THUS EVERY INTEGER $n > 1$ CAN BE WRITTEN IN THIS FORM BY THE PSI.

⑤ PROVE THAT aRb IFF $[a] = [b]$, WHERE R IS AN EQUIVALENCE RELATION ON A.

PF $\Leftarrow$ ASSUME THAT $[a] = [b]$. SINCE bRb, $b \in [b]$ SO $b \in [a]$ AND THEREFORE aRb.

$\Rightarrow$ ASSUME THAT aRb.

1) IF $x \in [a]$, THEN aRx. SINCE aRb, bRa BY SYMMETRY SO bRx BY TRANSITIVITY. HENCE $x \in [b]$, SO $[a] \subseteq [b]$.

2) IF $x \in [b]$, THEN bRx. SINCE aRb, aRx BY TRANSITIVITY; SO $x \in [a]$ AND THEREFORE $[b] \subseteq [a]$.

THUS $[a] = [b]$.

MATH 108 ANSWERS – QUIZ 2B

① $P = \{\{a\}, \{b\}, \{d\}, \{c,e\}\}$

$R = \{(a,a), (b,b), (d,d), (c,c), (c,e), (e,c), (e,e)\}$

② EVERY INTEGER $n \geq 18$ CAN BE WRITTEN AS $n = 4a + 7b$ WHERE $a, b \in \mathbb{Z}$ WITH $a, b \geq 0$.

PF 1) ANY INTEGER $k \in \{18, 19, 20, 21\}$ CAN BE WRITTEN IN THIS FORM SINCE $18 = 4(1) + 7(2)$, $19 = 4(3) + 7(1)$, $20 = 4(5) + 7(0)$, $21 = 4(0) + 7(3)$.

2) LET $n \in \mathbb{Z}$ WITH $n \geq 21$, AND ASSUME THAT ANY INTEGER k WITH $18 \leq k \leq n$ CAN BE WRITTEN IN THIS FORM.

THEN $n - 3 = 4a + 7b$ FOR SOME INTEGERS $a, b \geq 0$,

SO $n + 1 = 4(a+1) + 7b$.

THUS ANY INTEGER $n \geq 18$ CAN BE WRITTEN IN THIS FORM BY THE PSI.

$\boxed{\text{OR}}$ PF 1) THIS IS TRUE FOR $n = 18$, SINCE $18 = 4(1) + 7(2)$.

2) ASSUME THAT n IS AN INTEGER WITH $n \geq 18$ SUCH THAT $n = 4a + 7b$ FOR SOME INTEGERS $a, b \geq 0$.

i) IF $b \geq 1$, THEN $n + 1 = 4(a+2) + 7(b-1)$ WITH $a + 2 \geq 0$ AND $b - 1 \geq 0$.

ii) IF $b = 0$, THEN $n = 4a \geq 18 \Rightarrow a \geq 5$, SO $n + 1 = 4(a-5) + 7(b+3)$ WITH $a - 5 \geq 0$ AND $b + 3 \geq 0$.

THEREFORE ANY INTEGER $n \geq 18$ CAN BE WRITTEN IN THIS FORM BY THE PMI.

③ A) RELATION ON $\mathbb{R}$ WHICH IS SYMMETRIC, TRANSITIVE, NOT REFLEXIVE

LET xRy IFF $x \geq 0$ AND $y \geq 0$.

(THIS IS NOT REFLEXIVE SINCE $-1 \not R -1$.)

$\boxed{\text{OR}}$ LET xRy IFF $xy > 0$.

(THIS IS NOT REFLEXIVE SINCE $0 \not R 0$.)

B) RELATION ON $\mathbb{R}$ WHICH IS REFLEXIVE, SYMMETRIC, AND NOT TRANSITIVE

LET xRy IFF $xy \geq 0$.

(THIS IS NOT TRANSITIVE SINCE $1 R 0$ AND $0 R -1$ BUT $1 \not R -1$.)

$\boxed{\text{OR}}$ LET xRy IFF $xy < 0$ OR $x = y$.

(THIS IS NOT TRANSITIVE SINCE $1 R -2$ AND $-2 R 3$ BUT $1 \not R 3$.)

MATH 108 ANSWERS – QUIZ 2B

(4) aRb IFF $5 \mid (a+4b)$, ON $\mathbb{Z}$.

1) R IS REFLEXIVE: IF $a \in \mathbb{Z}$, THEN aRa SINCE $5 \mid (5a)$.

2) R IS SYMMETRIC:

LET aRb, SO $5 \mid (a+4b)$ AND THEREFORE $a+4b = 5k$ FOR SOME $k \in \mathbb{Z}$.

THEN $b+4a = (5b+5a) - (a+4b) = (5b+5a) - 5k = 5(b+a-k)$

WITH $b+a-k \in \mathbb{Z}$, SO $5 \mid (b+4a)$ AND HENCE bRa.

3) R IS TRANSITIVE:

LET aRb AND bRc, SO $5 \mid (a+4b)$ AND $5 \mid (b+4c)$.

THEN $a+4b = 5k$ AND $b+4c = 5\ell$ FOR SOME $k, \ell \in \mathbb{Z}$,

SO $a+4c = (a+4b) + (b+4c) - 5b = 5k + 5\ell - 5b = 5(k+\ell-b)$

WITH $k+\ell-b \in \mathbb{Z}$. THEREFORE $5 \mid (a+4c)$, SO aRc.

THEREFORE R IS AN EQUIVALENCE RELATION ON $\mathbb{Z}$.

(5) IF $E_c = \{(x,y) \in \mathbb{R}^2 : x = 5y^2 + c\}$,

THEN $\{E_c : c \in \mathbb{R}\}$ IS A PARTITION OF $\mathbb{R} \times \mathbb{R}$.

PF 1) IF $c \in \mathbb{R}$, THEN $(c,0) \in E_c$; SO $E_c \neq \emptyset$.

2) LET $(x,y) \in \mathbb{R} \times \mathbb{R}$, AND LET $c = x - 5y^2$.

THEN $x = 5y^2 + c$, SO $(x,y) \in E_c$.

THEREFORE $\mathbb{R} \times \mathbb{R} \subseteq \bigcup_{c \in \mathbb{R}} E_c$, SO $\bigcup_{c \in \mathbb{R}} E_c = \mathbb{R} \times \mathbb{R}$ (SINCE $\bigcup_{c \in \mathbb{R}} E_c \subseteq \mathbb{R} \times \mathbb{R}$).

3) IF $(x,y) \in E_c \cap E_d$,

THEN $x = 5y^2 + c$ AND $x = 5y^2 + d$;

SO $c = x - 5y^2 = d$ AND THUS $E_c = E_d$.

THEREFORE IF $E_c \neq E_d$, THEN $E_c \cap E_d = \emptyset$.

THUS $\{E_c : c \in \mathbb{R}\}$ IS A PARTITION OF $\mathbb{R} \times \mathbb{R}$.