MATH 108 ANSWERS - QUIZ 3A

① $f : (2,\infty) \to (8,\infty)$ with $f(x) = \dfrac{8x}{x-2}$.

A) If $f(x) = f(y)$, then

$$\frac{8x}{x-2} = \frac{8y}{y-2} \Rightarrow 8x(y-2) = 8y(x-2) \Rightarrow 8xy - 16x = 8xy - 16y \Rightarrow 16y = 16x \Rightarrow x = y,$$

so f is 1-1.

(or show that $f'(x) = \dfrac{-16}{(x-2)^2} < 0$ on $(2,\infty)$, so f is decreasing and therefore 1-1.)

B) Let $y \in (8,\infty)$, and let $x = \dfrac{2y}{y-8}$.

Then $x \in (2,\infty)$ since $x - 2 = \dfrac{2y}{y-8} - 2 = \dfrac{16}{y-8} > 0$ since $y > 8$,

and $f(x) = \dfrac{8x}{x-2} = \dfrac{\frac{16y}{y-8}}{\frac{16}{y-8}} = \dfrac{16y}{16} = y$.

Therefore $y \in \text{Rng}(f)$, so $\text{Rng}(f) = (8,\infty)$.

② f from $\mathbb{Z}_{10}$ to $\mathbb{Z}_6$ given by $f(\overline{x}) = [3x-2]$ is a well-defined function.

PF Let $\overline{x} = \overline{y}$ in $\mathbb{Z}_{10}$, so $x \equiv y \pmod{10}$ and therefore $10 \mid (x-y)$.

Then $x - y = 10k$ for some $k \in \mathbb{Z}$, and

$(3x-2) - (3y-2) = 3(x-y) = 3(10k) = 6(5k)$ where $5k \in \mathbb{Z}$,

therefore $3x - 2 \equiv 3y - 2 \pmod 6$ since $6 \mid ((3x-2) - (3y-2))$,

so $[3x-2] = [3y-2]$ in $\mathbb{Z}_6$ and thus $f(\overline{x}) = f(\overline{y})$.

③ Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $f(m,n) = 2^{m-1}(2n-1)$.

④ If $f : B \to C$ and $g : A \to B$ and $f \circ g$ is 1-1, then g is 1-1.

PF Let $g(a) = g(e)$ where $a, e \in A$.

Then $f(g(a)) = f(g(e))$,

so $(f \circ g)(a) = (f \circ g)(e)$ and

therefore $a = e$ since $f \circ g$ is 1-1.

Thus g is 1-1.

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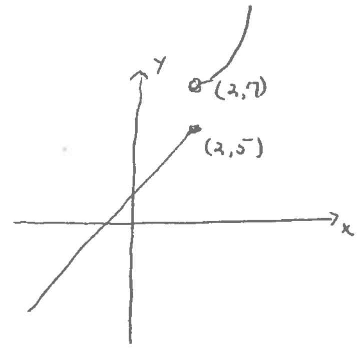


(5) $f:\mathbb{R}\to\mathbb{R}$ WITH $f(x)=\begin{cases} x+3, & \text{IF } x\le 2 \\ x^2+3, & \text{IF } x>2. \end{cases}$

A) LET $f(a)=f(b)$.

1) IF $a\le 2$ AND $b\le 2$, THEN $a+3=b+3$ SO $a=b$.

2) IF $a>2$ AND $b>2$, THEN $a^2+3=b^2+3$ SO $a^2=b^2$ AND HENCE $a=b$ (SINCE $a,b>0$).

3) IF $a\le 2$ AND $b>2$, THEN $a+3=b^2+3$ WITH $a+3\le 5$ AND $b^2+3>7$; SO THIS GIVES A CONTRADICTION.

4) IF $a>2$ AND $b\le 2$, INTERCHANGING THE ROLES OF a AND b IN 3) GIVES A CONTRADICTION.

THEREFORE $f(a)=f(b)\Rightarrow a=b$, SO f IS 1-1.

OR i) IF $x\le 2$, THEN $x+3\le 5$ SO $f(x)\le 5$, AND THEREFORE $f(x)>5\Rightarrow x>2$.

ii) IF $x>2$, THEN $f(x)=x^2+3>7$, AND THEREFORE $f(x)\le 7\Rightarrow x\le 2$.

LET $f(a)=f(b)$.

1) IF $f(a)>5$, THEN $a,b>2$ SO $a^2+3=b^2+3\Rightarrow a^2=b^2\Rightarrow a=b$ (SINCE $a,b>0$).

2) IF $f(a)\le 5$, THEN $a,b\le 2$ SO $a+3=b+3\Rightarrow a=b$.

SINCE $f(a)=f(b)\Rightarrow a=b$, f IS 1-1.

B) ASSUME THAT $6\in Rng(f)$, SO $f(x)=6$ FOR SOME $x\in\mathbb{R}$.

1) IF $x\le 2$, THEN $f(x)=x+3=6$ SO $x=3$, WHICH GIVES A CONTRADICTION.

2) IF $x>2$, THEN $f(x)=x^2+3=6$ SO $x^2=3$ AND $x=\sqrt{3}$, WHICH GIVES A CONTRADICTION.

THEREFORE $6\notin Rng(f)$, SO f IS NOT ONTO.

OR 1) IF $x\le 2$, THEN $f(x)=x+3\le 5$

2) IF $x>2$, THEN $f(x)=x^2+3>7$.

THEREFORE $Rng(f)\subseteq(-\infty,5]\cup(7,\infty)$;

SO $Rng(f)\ne\mathbb{R}$ AND THEREFORE f IS NOT ONTO.

MATH 108 ANSWERS – QUIZ 3B

① $f:(3,\infty)\to(-\infty,-5)$ WITH $f(x)=\dfrac{5x}{3-x}$.

A) IF $\underline{f(x)=f(y)}$, THEN

$$\frac{5x}{3-x}=\frac{5y}{3-y} \implies 5x(3-y)=5y(3-x) \implies 15x-5xy=15y-5xy \implies 15x=15y \implies \underline{x=y},$$

SO f IS 1-1.

(OR SHOW THAT $f'(x)=\dfrac{15}{(3-x)^2}>0$ ON $(3,\infty)$, SO f IS INCREASING AND THEREFORE 1-1.)

B) LET $\underline{y\in(-\infty,-5)}$, AND LET $\underline{x=\dfrac{3y}{y+5}}$.

THEN $\underline{x\in(3,\infty)}$ SINCE $x-3=\dfrac{3y}{y+5}-3=-\dfrac{15}{y+5}>0$ SINCE $y+5<0$,

AND $\underline{f(x)}=\dfrac{5x}{3-x}=\dfrac{\frac{15y}{y+5}}{\frac{15}{y+5}}=\dfrac{15y}{15}=\underline{y}$.

THEREFORE $y\in Rng(f)$, SO $Rng(f)=(-\infty,-5)$.

② f FROM $\mathbb{Z}_6$ TO $\mathbb{Z}_8$ GIVEN BY $f(\bar{x})=[4x+5]$ IS A WELL-DEFINED FUNCTION.

PF LET $\underline{\bar{x}=\bar{y}}$ IN $\mathbb{Z}_6$, SO $\underline{x\equiv y \pmod 6}$ AND THEREFORE $6\mid(x-y)$.

THEN $x-y=6k$ FOR SOME k IN $\mathbb{Z}$, AND

$(4x+5)-(4y+5)=4x-4y=4(x-y)=4(6k)=8(3k)$ WHERE $3k\in\mathbb{Z}$.

THEREFORE $\underline{4x+5\equiv 4y+5 \pmod 8}$ SINCE $8\mid((4x+5)-(4y+5))$,

SO $[4x+5]=[4y+5]$ IN $\mathbb{Z}_8$ AND THUS $\underline{f(\bar{x})=f(\bar{y})}$.

③ LET $f:\mathbb{Z}\to\mathbb{N}$ WITH $f(n)=\begin{cases}2n, & \text{IF } n>0\\ 1-2n, & \text{IF } n\le 0.\end{cases}$

$$\begin{array}{ccccccccc} \mathbb{Z}=\{ & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \dots \}\\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbb{N}=\{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \dots \}\end{array}$$

④ IF $g:C\to D$ AND $h:D\to E$ ARE 1-1, THEN $h\circ g$ IS 1-1.

PF LET $\underline{(h\circ g)(a)=(h\circ g)(b)}$ WHERE $a,b\in C$.

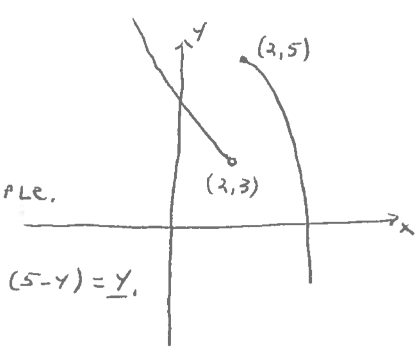
THEN $h(g(a))=h(g(b))$,

SO $g(a)=g(b)$ SINCE h IS 1-1

AND $\underline{a=b}$ SINCE g IS 1-1.

THEREFORE $h\circ g$ IS 1-1.

⑤ $f:\mathbb{R}\to\mathbb{R}$ WITH $f(x)=\begin{cases}5-x, & \text{IF } x<2.\\ 9-x^2, & \text{IF } x\ge 2.\end{cases}$



A) f IS NOT 1-1 SINCE $\underline{f(0)}=5=\underline{f(2)}$, FOR EXAMPLE.

B) LET $y\in\mathbb{R}$,

1) IF $y>3$, LET $x=5-y$. THEN $x<2$, AND $\underline{f(x)}=5-(5-y)=\underline{y}$.

2) IF $y\le 3$, THEN $9-y\ge 6$ SO LET $x=\sqrt{9-y}$.

THEN $x\ge\sqrt{6}>2$, AND $\underline{f(x)}=9-x^2=9-(9-y)=\underline{y}$.

THEREFORE $y\in Rng(f)$ IN BOTH CASES, SO $Rng(f)=\mathbb{R}$ AND f IS ONTO.