MATH 108 SOLUTIONS – QUIZ 4

1.

C	A) $(9,\infty)$	C	D) $\mathbb{R}-\mathbb{Q}$	C	G) $\mathcal{P}(\mathbb{N})$	A) $\aleph_0$	D) 8
C	B) $(2,8)$	A	E) $\mathbb{Q}-\mathbb{Z}$	E	H) $\mathcal{P}(\mathbb{R})$	B) 6	E) NONE OF THE ABOVE
A	C) $\mathbb{N}\times\mathbb{N}$	D	F) $\mathcal{P}(\{x,y,z\})$	C	I) $\mathbb{R}\times\mathbb{R}$	C) c	

2. DEFINE A BIJECTION $f:(0,1)\to(-\infty,5)$, AND SHOW THAT f MAPS INTO $(-\infty,5)$ AND IS BIJECTIVE.

LET $f(x) = \ln x + 5$.

1) IF $0<x<1$, THEN $\ln x<0$ SO $\ln x+5<5$ AND THEREFORE $f(x)\in(-\infty,5)$.

2) IF $f(x)=f(y)$, THEN $\ln x+5=\ln y+5$ SO $\ln x=\ln y$ AND HENCE $x=y$. THEREFORE f IS 1-1.

3) IF $y\in(-\infty,5)$, LET $x=e^{y-5}$. THEN $x>0$, AND $x<1$ SINCE $y-5<0$ SO $e^{y-5}<1$; AND THEREFORE $x\in(0,1)$ WITH $f(x)=\ln x+5=(y-5)+5=y$. THUS $y\in \mathrm{Rng}(f)$, SO f IS ONTO.

(OR USE $f(x)=6-\frac{1}{x}$ INSTEAD.)

3. IF T IS THE SET OF ALL INFINITE SEQUENCES OF 0'S AND 1'S WHICH HAVE ONLY FINITELY MANY 1'S, SHOW THAT T IS DENUMERABLE.

PF DEFINE $f:T\to\mathbb{R}$ BY $f(t_1t_2t_3\cdots) = .t_1t_2t_3\cdots$

THEN $f(t)\in\mathbb{Q}$ FOR EVERY $t\in T$ SINCE $.t_1t_2t_3\cdots$ IS A FINITE DECIMAL, AND f IS 1-1 SINCE $f(t_1t_2t_3\cdots)=f(u_1u_2u_3\cdots)\implies .t_1t_2t_3\cdots = .u_1u_2u_3\cdots$
$\implies t_n=u_n$ FOR ALL $n \implies t_1t_2t_3\cdots = u_1u_2u_3\cdots$

THEREFORE $T\approx f(T)$, AND $f(T)$ IS DENUMERABLE SINCE IT IS AN INFINITE SUBSET OF THE DENUMERABLE SET $\mathbb{Q}$; SO T IS DENUMERABLE.

(OR) PF LET $p_1,p_2,p_3,\ldots$ BE THE SET OF PRIMES LISTED IN ORDER, AND DEFINE $f:T\to\mathbb{N}$ BY $f(t_1t_2t_3\cdots)=p_1^{t_1}p_2^{t_2}p_3^{t_3}\cdots$.

(NOTICE THAT $f(t)\in\mathbb{N}$ FOR EVERY $t\in T$, SINCE $t_n\neq 0$ FOR ONLY FINITELY MANY n.)

THEN f IS 1-1 SINCE $f(t_1t_2t_3\cdots)=f(u_1u_2u_3\cdots)\implies$
$p_1^{t_1}p_2^{t_2}p_3^{t_3}\cdots = p_1^{u_1}p_2^{u_2}p_3^{u_3}\cdots \implies u_n=t_n$ FOR ALL n (BY THE FUNDAMENTAL TH. OF ARITHMETIC) $\implies t_1t_2t_3\cdots = u_1u_2u_3\cdots$,

SO $T\approx f(T)$. SINCE $f(T)$ IS AN INFINITE SUBSET OF $\mathbb{N}$, $f(T)$ IS DENUMERABLE; SO T IS DENUMERABLE.

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4 A) SHOW THAT $(1,9) \approx [1,9]$.

PF 1) $(1,9) \subseteq [1,9]$, SO $|(1,9)| \le |[1,9]|$

2) $[1,9] \subseteq (0,10)$, SO $|[1,9]| \le |(0,10)|$ WHERE $|(0,10)| = |(1,9)|$

SINCE $(0,10) \approx (1,9)$ AND THEREFORE $|[1,9]| \le |(1,9)|$.

OR DEFINE $f: [1,9] \to (1,9)$ BY $f(x) = \frac{1}{2}(x+5)$.

THEN f IS 1-1 SINCE $f(x) = f(y) \Rightarrow \frac{1}{2}(x+5) = \frac{1}{2}(y+5) \Rightarrow x+5 = y+5 \Rightarrow x = y$,

SO $|[1,9]| \le |(1,9)|$.

BY THE CSB TH., $|(1,9)| = |[1,9]|$ SO $(1,9) \approx [1,9]$.

B) SHOW THAT $(0,1) \times (0,1) \approx (0,1)$.

1) DEFINE $f: (0,1) \to (0,1) \times (0,1)$ BY $f(x) = (x,x)$.

THEN f IS 1-1 SINCE $f(x) = f(y) \Rightarrow (x,x) = (y,y) \Rightarrow x = y$,

SO $|(0,1)| \le |(0,1) \times (0,1)|$.

2) DEFINE $g: (0,1) \times (0,1) \to (0,1)$ BY

$g(x,y) = .x_1 y_1 x_2 y_2 x_3 y_3 \cdots$ IF $x = .x_1 x_2 x_3 \cdots$ AND $y = .y_1 y_2 y_3 \cdots$

THEN g IS 1-1 SINCE

$g(x,y) = g(a,b) \Rightarrow .x_1 y_1 x_2 y_2 x_3 y_3 \cdots = .a_1 b_1 a_2 b_2 a_3 b_3 \cdots$

$\Rightarrow .x_1 x_2 x_3 \cdots = .a_1 a_2 a_3 \cdots$ AND $.y_1 y_2 y_3 \cdots = .b_1 b_2 b_3 \cdots$

$\Rightarrow x = a$ AND $y = b \Rightarrow (x,y) = (a,b)$.

THEREFORE $|(0,1) \times (0,1)| \le |(0,1)|$,

BY THE CSB TH., $|(0,1) \times (0,1)| = |(0,1)|$ SO $(0,1) \times (0,1) \approx (0,1)$.