MATH 108 ANSWERS – TEST 1 REVIEW PROBLEMS

① A) $p \Rightarrow q \equiv (\sim p) \vee q$

B) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

C) $\sim (p \wedge q) \equiv p \Rightarrow (\sim q)$

D) $p \Rightarrow (q \vee r) \equiv (p \wedge (\sim q)) \Rightarrow r$

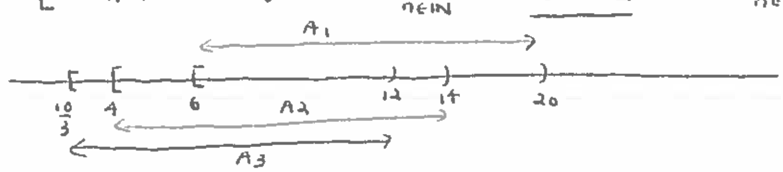
② PF Let $a|b$ and $a|c$, so $b = ka$ and $c = la$ for some integers k and l.
Then $b + c = ka + la = (k+l)a$ where $k + l \in \mathbb{Z}$, so $a|(b+c)$.

③ PF (of the contrapositive)
If n is odd, then $n = 2k+1$ for some $k \in \mathbb{Z}$.
Then $(n+5)^2 = (2k+6)^2 = 4k^2 + 24k + 36 = 2(2k^2 + 12k + 18)$
where $2k^2 + 12k + 18 \in \mathbb{Z}$, so $(n+5)^2$ is even.

④ $\sim (\forall \varepsilon)\{\varepsilon > 0 \Rightarrow [(\exists \delta)(\delta > 0 \wedge (\forall x)(0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon))]\}$
$\equiv (\exists \varepsilon)\{\varepsilon > 0 \wedge [(\forall \delta)(\delta > 0 \Rightarrow (\exists x)(0 < |x-a| < \delta \wedge |f(x) - L| \geq \varepsilon))]\}$

⑤ PF (by contradiction)
Let $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, and assume instead that $x + y \in \mathbb{Q}$.
Then $x = \frac{m}{n}$ and $x + y = \frac{p}{q}$ where $m, n, p, q \in \mathbb{Z}$ and $n, q \neq 0$.
Therefore $y = (x+y) - x = \frac{p}{q} - \frac{m}{n} = \frac{pn - qm}{qn}$ where
$pn - qm$ and qn are integers with $qn \neq 0$, so $y \in \mathbb{Q}$.
This gives a contradiction; so if $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $x + y \notin \mathbb{Q}$.

⑥ $A_n = \left[2 + \frac{4}{n}, 8 + \frac{12}{n}\right)$ a) $\bigcap_{n \in \mathbb{N}} A_n = [6, 8]$ b) $\bigcup_{n \in \mathbb{N}} A_n = (2, 20)$



⑦ Prove that if $A \cup B \subseteq C \cup D$, $A \cap B = \emptyset$, and $D \subseteq B$, then $A \subseteq C$.
PF Let $x \in A$. Then $x \in A \cup B$, so $x \in C \cup D$ since $A \cup B \subseteq C \cup D$.
Thus either $x \in C$ or $x \in D$.
If $x \in D$, then $x \in B$ since $D \subseteq B$; so $x \in A$ and $x \in B$
implies that $x \in A \cap B = \emptyset$, which is impossible.
Therefore $x \notin D$, so $x \in C$.
Hence $A \subseteq C$.

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(8) PROVE THAT $3^n > n^3$ FOR ALL INTEGERS $n \geq 4$ USING THE PMI.

PF 1) THIS IS TRUE FOR $n = 4$, SINCE $3^4 = 81 > 64 = 4^3$.

2) LET $n \in \mathbb{Z}$ WITH $n \geq 4$ SUCH THAT $3^n > n^3$.

THEN $\underline{3^{n+1}} = 3(3^n) > 3n^3 = n^3 + 2n^3 \geq n^3 + 8n^3 = n^3 + 3n^2 + 5n^3$

$\geq n^3 + 3n^2 + 20n = n^3 + 3n^2 + 3n + 17n > n^3 + 3n^2 + 3n + 1 = \underline{(n+1)^3}$.

THEREFORE $3^n > n^3$ FOR ALL INTEGERS $n \geq 4$ BY THE PMI.

$\boxed{\text{OR}}$ USE 2) LET $n \in \mathbb{Z}$ WITH $3^n > n^3$ AND $n \geq 4$.

THEN $\underline{3^{n+1}} = 3(3^n) > \underline{3n^3}$; AND SINCE $n \geq 4$, $\frac{1}{n} \leq \frac{1}{4}$ SO

$\frac{(n+1)^3}{n^3} = \left(\frac{n+1}{n}\right)^3 = \left(1 + \frac{1}{n}\right)^3 \leq \left(1 + \frac{1}{4}\right)^3 = \left(\frac{5}{4}\right)^3 = \frac{125}{64} < 3$.

THEREFORE $\underline{3n^3 > (n+1)^3}$, SO $\underline{3^{n+1} > (n+1)^3}$.

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(9) PROVE THAT $\sqrt[5]{3}$ IS IRRATIONAL.

PF (BY CONTRADICTION)

ASSUME INSTEAD THAT $\sqrt[5]{3}$ IS RATIONAL, SO $\sqrt[5]{3} = \frac{m}{n}$ WHERE $m, n \in \mathbb{Z}$ WITH $n \neq 0$ AND $\frac{m}{n}$ IN REDUCED FORM.

THEN $3 = \frac{m^5}{n^5}$, SO $\underline{m^5 = 3n^5}$.

SINCE $n^5 \in \mathbb{Z}$, $3 \mid m^5$ AND THEREFORE $\underline{3 \mid m}$.

THEN $\underline{m = 3k}$ FOR SOME $k \in \mathbb{Z}$, SO $(3k)^5 = 3^5k^5 = 3n^5$ AND HENCE $\underline{81k^5 = n^5}$.

SINCE $3(27k^5) = n^5$ AND $27k^5 \in \mathbb{Z}$, $3 \mid n^5$ AND THEREFORE $\underline{3 \mid n}$.

THIS GIVES A CONTRADICTION, SINCE 3 IS A COMMON DIVISOR OF m AND n;

SO $\sqrt[5]{3}$ IS IRRATIONAL.

$\boxed{\text{OR}}$ PF (BY CONTRADICTION)

ASSUME INSTEAD THAT $\sqrt[5]{3}$ IS RATIONAL, SO $\sqrt[5]{3} = \frac{m}{n}$ WHERE $m \in \mathbb{Z}$, $n \in \mathbb{N}$, AND $\frac{m}{n}$ IS IN REDUCED FORM.

THEN $3 = \frac{m^5}{n^5}$, SO $\underline{m^5 = 3n^5}$.

$\underline{\text{IF } n > 1}$, LET p BE A PRIME DIVISOR OF n.

SINCE $p \mid n$ AND $n \mid 3n^5$, $p \mid 3n^5$; SO $p \mid m^5$ AND THEREFORE $p \mid m$.

THIS CONTRADICTS THE FACT THAT m AND n HAVE NO COMMON DIVISORS GREATER THAN 1,

SO $\underline{n = 1}$ AND $\underline{m^5 = 3}$.

THIS GIVES A CONTRADICTION SINCE $1^5 < 3 < 2^5$, SO $\sqrt[5]{3}$ IS IRRATIONAL.

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(10) PF (USING THE PSI)

1) THIS IS TRUE FOR $24 \le n \le 27$, SINCE
$24 = 4(6)+9(0)$, $25 = 4(4)+9(1)$, $26 = 4(2)+9(2)$, AND $27 = 4(0)+9(3)$.

2) ASSUME THAT n IS AN INTEGER WITH $n \ge 27$ SUCH THAT THE ASSERTION HOLDS FOR ALL INTEGERS i WITH $24 \le i \le n$.
THEN $24 \le n-3 \le n$, SO $n-3 = 4a+9b$ FOR SOME INTEGERS $a, b \ge 0$
AND THEREFORE $n+1 = 4(a+1)+9b$ WITH $a+1, b \ge 0$.

THUS THE ASSERTION HOLDS FOR ANY INTEGER $n \ge 24$ BY THE PSI.

$\boxed{\text{OR}}$ PF (USING THE PMI)

1) THIS IS TRUE FOR $n = 24$, SINCE $24 = 4(6)+9(0)$.

2) ASSUME THAT n IS AN INTEGER WITH $n \ge 24$ SUCH THAT
$n = 4a+9b$ FOR SOME INTEGERS $a, b \ge 0$.

A) IF $a \ge 2$, THEN $n+1 = 4(a-2)+9(b+1)$ WITH $a-2, b+1 \ge 0$.

B) IF $a \le 1$, THEN $24 \le n \le 4+9b$ SO $b \ge 3$
AND $n+1 = 4(a+7)+9(b-3)$ WITH $a+7, b-3 \ge 0$.

THEREFORE $n+1$ CAN BE WRITTEN IN THIS FORM IN EITHER CASE;
SO THE ASSERTION HOLDS FOR ANY INTEGER $n \ge 24$ BY THE PMI.

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(11) PF (BY CONTRADICTION)

LET S BE A SUBSET OF $\mathbb{N}$ SUCH THAT
1) $1 \in S$ AND 2) $\forall n \in \mathbb{N}$, $n \in S \Rightarrow n+1 \in S$;
AND ASSUME THAT $S \ne \mathbb{N}$.

THEN $T = \mathbb{N} - S \ne \emptyset$, SO T HAS A LEAST ELEMENT m BY THE WOP.

THEN $m \ne 1$ SINCE $1 \in S$,
SO $m-1 \in \mathbb{N}$ WITH $m-1 \notin T$ SO $m-1 \in S$.
THEREFORE $m = (m-1)+1 \in S$, AND THIS GIVES A CONTRADICTION
SINCE $m \in T = \mathbb{N} - S$.

THEREFORE $S = \mathbb{N}$.