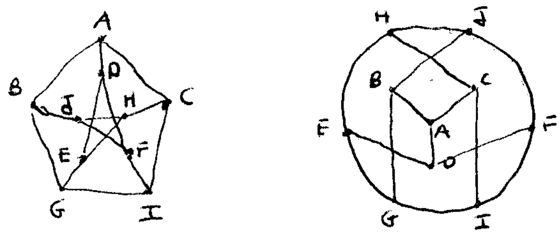
MATH 145

CH. 11 – SOLUTIONS

(12) SINCE THE 3RD GRAPH HAS CYCLES OF LENGTH 4 AND THE OTHER GRAPHS DO NOT, THE 3RD GRAPH IS NOT ISOMORPHIC TO THE FIRST TWO.

THE FOLLOWING GIVES AN ISOMORPHISM BETWEEN THE FIRST TWO GRAPHS:



(42) IF G IS A GRAPH OF ORDER n SUCH THAT $\deg(x)+\deg(y) \geq n-1$ WHENEVER x AND y ARE NOT ADJACENT VERTICES, THEN G HAS A HAMILTON PATH.

PF WE CAN ASSUME THAT $n \geq 2$, SO LET H BE THE GRAPH OBTAINED FROM G BY ADDING A NEW VERTEX z AND CONNECTING z WITH EVERY VERTEX OF G WITH AN EDGE. THEN H HAS $n+1$ VERTICES, AND EVERY VERTEX IN G HAS ITS DEGREE INCREASED BY ONE IN H; SO $\underline{\deg(x)+\deg(y) \geq n+1}$ WHENEVER x AND y ARE NON-ADJACENT VERTICES IN H. BY ORE'S THEOREM, H HAS A HAMILTON CYCLE $z v_1 \ldots v_n z$, SO $v_1 \ldots v_n$ IS A HAMILTON PATH IN G.

(44) A) $K_{m,n}$ HAS A HAMILTON CYCLE IFF $m=n$:

A BIPARTITE GRAPH WITH BIPARTITION A, B DOES NOT HAVE A HAMILTON CYCLE IF $|A| \neq |B|$, AND $K_{n,n}$ HAS THE HAMILTON CYCLE $a_1 b_1 a_2 b_2 \ldots a_n b_n a_1$.

B) $K_{m,n}$ HAS A HAMILTON PATH IFF $n-1 \leq m \leq n+1$:

A BIPARTITE GRAPH WITH BIPARTITION A, B DOES NOT HAVE A HAMILTON PATH IF $|A|$ AND $|B|$ DIFFER BY MORE THAN ONE, AND $K_{m,n}$ HAS A HAMILTON PATH IF $m=n-1$, $m=n$, OR $m=n+1$ GIVEN BY $b_1 a_1 b_2 a_2 \ldots b_{n-1} a_m b_n$, $a_1 b_1 a_2 b_2 \ldots a_n b_n$, OR $a_1 b_1 a_2 b_2 \ldots a_n b_n a_m$

(45) LET G BE A MULTIGRAPH WITH CONNECTED COMPONENTS $G_1, \ldots, G_k$. THEN G IS BIPARTITE IFF G_i IS BIPARTITE FOR EACH i.

PF $\Rightarrow$ IF G IS BIPARTITE, LET A, B BE A BIPARTITION OF G. IF WE LET $A_i = A \cap V(G_i)$ AND $B_i = B \cap V(G_i)$ FOR EACH i, THEN A_i, B_i GIVES A BIPARTITION OF G_i FOR EACH i (SINCE NO 2 ELEMENTS OF A_i ARE ADJACENT, AND SIMILARLY FOR B_i).

$\Leftarrow$ SUPPOSE G_i IS BIPARTITE FOR EACH i, WITH BIPARTITION A_i, B_i. IF $A = \bigcup_{i=1}^{k} A_i$ AND $B = \bigcup_{i=1}^{k} B_i$, THEN A, B IS A BIPARTITION OF G SINCE $V = A \cup B$ AND IF e IS AN EDGE OF G, THEN $e \in G_i$ FOR SOME i SO e CONNECTS A VERTEX IN A_i AND A VERTEX IN B_i (AND THEREFORE A VERTEX IN A AND A VERTEX IN B).

(47) IF G IS A BIPARTITE MULTIGRAPH WITH AN ODD NUMBER OF VERTICES, THEN G DOES NOT HAVE A HAMILTON CYCLE.

PF LET A, B BE A BIPARTITION OF G. IF G HAS A HAMILTON CYCLE, THEN $|A| = |B|$; SO G HAS AN EVEN NUMBER OF VERTICES.

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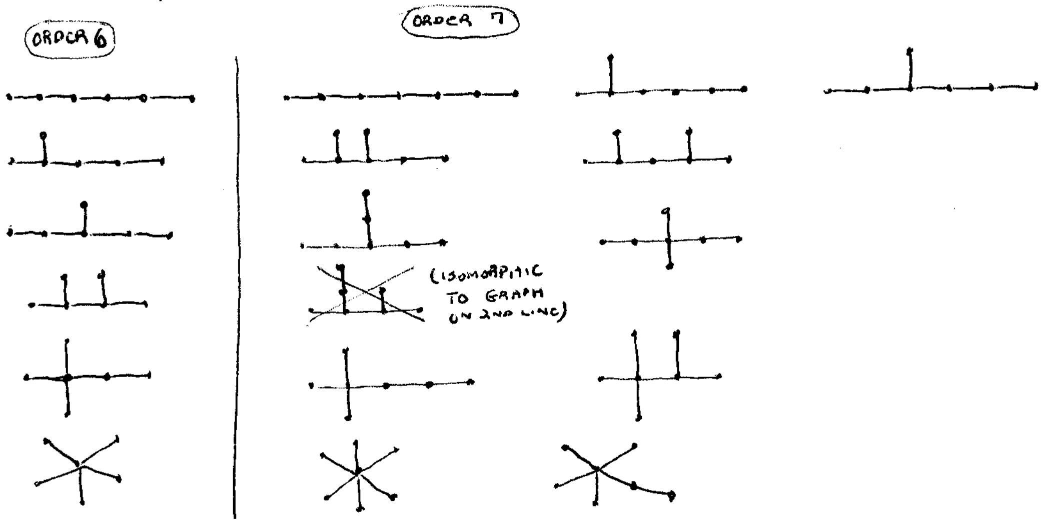
(53) A GRAPH IS A TREE IFF IT DOES NOT CONTAIN ANY CYCLES, BUT THE INSERTION OF ANY NEW EDGE ALWAYS CREATES EXACTLY ONE CYCLE.

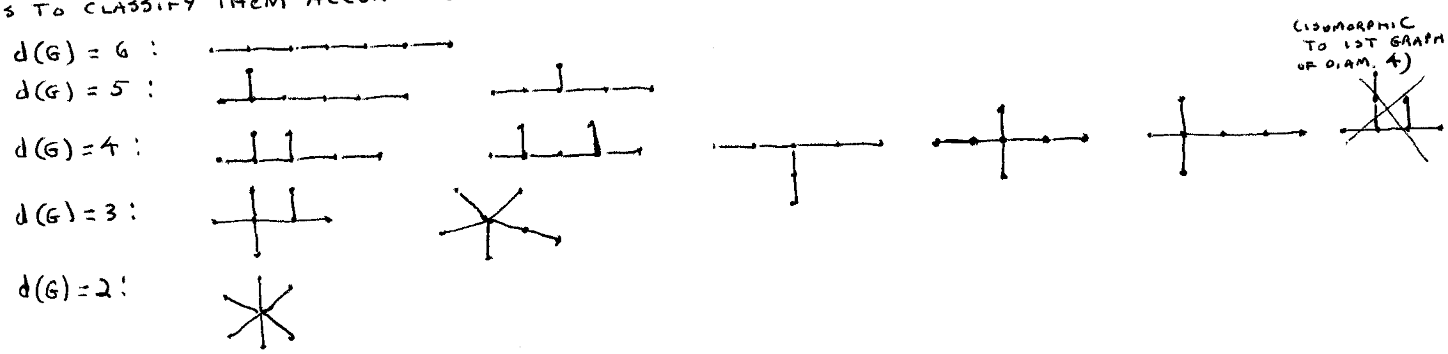
PF $\Rightarrow$ IF G IS A TREE, THEN IT DOES NOT CONTAIN ANY CYCLES. IF u AND v ARE ANY VERTICES IN G THAT ARE NOT ADJACENT, THERE IS A PATH FROM u TO v SINCE G IS CONNECTED; SO INSERTION OF THE EDGE uv GIVES A CYCLE. IF INSERTION OF uv CREATED MORE THAN ONE CYCLE, THEN REMOVING uv FROM THE CYCLES WOULD GIVE 2 DISTINCT PATHS FROM u TO v; AND THIS IS IMPOSSIBLE SINCE G IS A TREE (BY TH. 11.5.5).

$\Leftarrow$ WE KNOW THAT G DOES NOT CONTAIN ANY CYCLES, SO LET u AND v BE 2 VERTICES IN G. IF u AND v ARE NOT ADJACENT, INSERTION OF THE EDGE uv GIVES A CYCLE; SO REMOVING uv FROM THE CYCLE RESULTS IN A PATH FROM u TO v. THEREFORE G IS CONNECTED, SO IT IS A TREE.

(56)



REMARK DAVID SAID THAT AN EASIER WAY OF DRAWING THE TREES OF ORDER 7, INSTEAD OF GROWING THEM FROM TREES OF ORDER 6 AS THE TEXT SUGGESTED, IS TO CLASSIFY THEM ACCORDING TO THEIR DIAMETER (MAX. DISTANCE BETWEEN VERTICES):



(62) LET $d_1, \dots, d_n$ BE THE DEGREES OF THE VERTICES,

SO $\underline{d_1 + \dots + d_n = 2e = 2(n-1)}$.

LET $d_n = P$, SO $\underline{d_1 + \dots + d_{n-1}} = 2(n-1) - P = \underline{2n-2-P}$.

LET K BE THE NUMBER OF LEAVES, AND LET $d_1 = \dots = d_K = 1$;

THEN $\underline{d_{K+1} + \dots + d_{n-1} = 2n-2-P-K}$.

WHERE $\underline{d_{K+1} + \dots + d_{n-1} \geq 2(n-K-1)}$ SINCE $d_i \geq 2$ FOR $K+1 \leq i \leq n-1$.

THEREFORE $2n-2-P-K \geq 2(n-K-1) = 2n-2K-2$, SO $\underline{K \geq P}$.