Since the 3rd graph has cycles of length 4 and the other graphs do not, the 3rd graph is not isomorphic to the first two.

The following gives an isomorphism between the first two graphs:

If \( G \) is a graph of order \( n \) such that \( \deg(x) + \deg(y) \geq n-1 \) whenever \( x \) and \( y \) are not adjacent vertices, then \( G \) has a Hamilton path.

**Proof:**
We can assume that \( n \geq 2 \), so let \( H \) be the graph obtained from \( G \) by
- adding a new vertex \( z \) and connecting \( z \) with every vertex of \( G \) with an edge.
Then \( H \) has \( n+1 \) vertices, and every vertex in \( G \) has its degree increased by one in \( H \); so \( \deg(x) + \deg(y) \geq n+1 \) whenever \( x \) and \( y \) are non-adjacent vertices in \( H \).
By Ore's Theorem, \( H \) has a Hamilton cycle \( z \nu_1 \ldots \nu_{n-1} z \), so \( \nu_1 \ldots \nu_n \) is a Hamilton path in \( G \).

**4.** \( K_{m,n} \) has a Hamilton cycle if \( m > n \).

A bipartite graph with bipartition \( A, B \) does not have a Hamilton cycle if \( |A| \neq |B| \), and \( K_{m,n} \) has the Hamilton cycle \( a_1 b_1 a_2 b_2 \ldots a_m b_m \).

**b)** \( K_{m,n} \) has a Hamilton path if \( n-1 \leq m \leq n+1 \):

A bipartite graph with bipartition \( A, B \) does not have a Hamilton path if \( |A| \neq |B| \) differ by more than one, and \( K_{m,n} \) has a Hamilton path if \( m = n-1 \), \( m = n+1 \), or \( m = n+1 \) is given by \( b_1 b_2 \ldots b_{n-1} a_n a_{n+1} \).

**4.3** Let \( G \) be a multigraph with connected components \( G_1, \ldots, G_k \).
Then \( G \) is bipartite iff \( G \) is bipartite for each \( i \).

**Proof**
- If \( G \) is bipartite, let \( A, B \) be a bipartition of \( G \).
- If we let \( A_i = A \cap V(G_i) \) and \( B_i = B \cap V(G_i) \) for each \( i \), then \( A_i, B_i \) gives a bipartition of \( G_i \) for each \( i \).
- (Since no 2 elements of \( A_i \) are adjacent, and similarly for \( B_i \).)

**Suppose** \( G_i \) is bipartite for each \( i \), with bipartition \( A_i, B_i \).
- If \( A = U A_i \) and \( B = U B_i \), then \( A, B \) is a bipartition of \( G \) since \( \forall i \exists e \in V \) and if \( e \) is an edge of \( G \), then \( e \in G_i \) for some \( i \), so \( e \) connects a vertex in \( A_i \) and a vertex in \( B_i \) (and therefore a vertex in \( A \) and a vertex in \( B \)).

**4.4** If \( G \) is a bipartite multigraph with an odd number of vertices, then \( G \) does not have a Hamilton cycle.

**Proof**
- Let \( A, B \) be a bipartition of \( G \). If \( G \) has a Hamilton cycle, then \( |A| = |B| \); so \( G \) has an even number of vertices.
53 A graph is a tree if it does not contain any cycles, but the insertion of any new edge always creates exactly one cycle.

\[
\begin{align*}
\text{PF} & \implies \text{If } G \text{ is a tree, then it does not contain any cycles.} \\
& \text{If } U \text{ and } V \text{ are any vertices in } G \text{ that are not adjacent, there is a path} \\
& \text{from } U \text{ to } V \text{ since } G \text{ is connected; so insertion of the edge } UV \text{ gives a cycle.} \\
& \text{If insertion of } UV \text{ created more than one cycle, then removing } UV \text{ from} \\
& \text{the cycle would give 2 distinct paths from } U \text{ to } V \text{; and this is impossible} \\
& \text{since } G \text{ is a tree (by Th. 11.5.5).}
\end{align*}
\]

\[
\begin{align*}
& \iff \text{We know that } G \text{ does not contain any cycles, so let } U \text{ and } V \text{ be 2 vertices in } G, \\
& \text{if } U \text{ and } V \text{ are not adjacent, insertion of the edge } UV \text{ gives a cycle;} \text{ so} \\
& \text{removing } UV \text{ from the cycle results in a path from } U \text{ to } V, \\
& \text{therefore } G \text{ is connected, so it is a tree.}
\end{align*}
\]

**Remark:** David said that an easier way of drawing the trees of order 7, instead of growing them from trees of order 6 as the text suggested, is to classify them according to their diameter (max. distance between vertices):

- \(d(G) = 6:\)
  \[
  \begin{array}{c}
  \includegraphics[width=\textwidth]{tree_6} \\
  \end{array}
  \]

- \(d(G) = 5:\)
  \[
  \begin{array}{c}
  \includegraphics[width=\textwidth]{tree_5} \\
  \end{array}
  \]

- \(d(G) = 4:\)
  \[
  \begin{array}{c}
  \includegraphics[width=\textwidth]{tree_4} \\
  \end{array}
  \]

- \(d(G) = 3:\)
  \[
  \begin{array}{c}
  \includegraphics[width=\textwidth]{tree_3} \\
  \end{array}
  \]

- \(d(G) = 2:\)
  \[
  \begin{array}{c}
  \includegraphics[width=\textwidth]{tree_2} \\
  \end{array}
  \]

63 Let \(d_1, \ldots, d_n\) be the degrees of the vertices,

so \(d_1 + \ldots + d_n = 2e = 2(n-1)\),

let \(d_n = p\), so \(d_1 + \ldots + d_{n-1} = 2(n-1) - p = 2n - 2 - p\).

Let \(k\) be the number of leaves, so \(d_1 = \ldots = d_k = 1\).

Then \(d_{k+1} + \ldots + d_{n-1} = 2n - 2 - p - k\),

where \(d_{k+1} + \ldots + d_{n-1} \geq 2(n-k-1)\) since \(d_i \geq 2\) for \(k+1 \leq i \leq n-1\),

therefore \(2n - 2 - p - k \geq 2(n-k-1) = 2n - 2k - 2\), so \(k \geq p\).