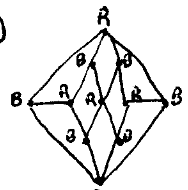
MATH 145 CH. 12 – SOLUTIONS

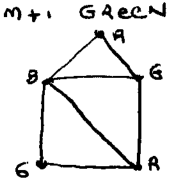
(4) Let C_n be an odd cycle $v_1 v_2 \cdots v_{2m+1} v_1$.

Then C_n cannot be colored with 2 colors, since then all odd vertices would have the same color and therefore v_1 and v_{2m+1} would be adjacent vertices with the same color (or because C_n is not bipartite).
If we color every even vertex red, every odd vertex blue except v_{2m+1}, and color v_{2m+1} green, we get a 3-coloring of C_n; so $\chi(C_n)=3$ for n odd.

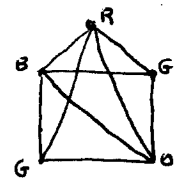
(5)



$\underline{\chi(G)=2}$



$\chi(G)\geq 3$, since G contains a triangle; so $\underline{\chi(G)=3}$ since G has a 3-coloring



($\chi(G)\geq 3$ since G contains a triangle, and) $\chi(G)\geq 4$ since G contains a copy of K_4; so $\underline{\chi(G)=4}$ since G has a 4-coloring

(6) Since $\chi(G)=k$, let H_i denote the vertices colored with the ith color for $1\leq i\leq k$, then each H_i is an independent set of vertices. If $H_i\cup H_j$ is an independent set for some $i\neq j$, then G could be colored with $k-1$ colors; so there is a vertex in H_i which is adjacent to a vertex in H_j for every $i\neq j$. Since there are $\binom{k}{2}$ pairs (i,j) with $1\leq i<j\leq k$, G has at least $\binom{k}{2}$ edges.

(9) $P_G(k)=k(k-1)^{n-1}=k\left(k^{n-1}-(n-1)k^{n-2}+\cdots+(-1)^{n-1}\right)=k^n-(n-1)k^{n-2}+\cdots+(-1)^{n-1}k.$

G is a connected graph (by the last term) and it has $n-1$ edges (from the 2nd term), so it is a tree since it has order n.

(10) We can color the endpoints of the deleted edge with the same color, and then we must use $n-2$ additional colors for the other vertices; so the chromatic number is $n-1$.

(11) $\underline{P_{K_n-e}(k)}=P_{K_n}(k)+P_{(K_n)_e}(k)=P_{K_n}(k)+P_{K_{n-1}}(k)=[k]_n+[k]_{n-1}.$

(13) We can use the same color for the endpoints of each of the deleted edges, and we need $n-4$ additional colors for the remaining vertices; so the chromatic number is $n-2$.

(14) $P_{C_n}(k)=(k-1)^n+(-1)^n(k-1)$

$\underline{\text{PF}}$ by induction on n:

1) If $n=3$, $\underline{P_{C_3}(k)}=(k-1)^3+(-1)^3(k-1)=(k-1)\left[(k-1)^2-1\right]=(k-1)\left[k^2-2k\right]=\underline{k(k-1)(k-2)}$.

2) Assume that $P_{C_n}(k)=(k-1)^n+(-1)^n(k-1)$ where $n\geq 3$.

Then $P_{C_{n+1}}(k)=P_{(C_{n+1})-e}(k)-P_{(C_{n+1})_e}(k)=P_{T_{n+1}}(k)-P_{C_n}(k)$

$=k(k-1)^{(n+1)-1}-\left[(k-1)^n+(-1)^n(k-1)\right]$

$=k(k-1)^n-(k-1)^n+(-1)^{n+1}(k-1)=(k-1)^{n+1}+(-1)^{n+1}(k-1)$,

so the formula is valid for $n+1$.

(15) Since the graph has an odd cycle, it is not bipartite; so it requires more than 2 colors. If we delete one edge in the cycle, then we get a bipartite graph, which requires only 2 colors; so we can color the graph with 3 colors by changing the color of one of the vertices of the deleted edge to a new color. Therefore its chromatic number is 3.