MATH 145

SOLUTIONS – CH. 12 AND GOLD SHEET

CH. 12 – (23) LET a_i BE THE NUMBER OF EDGES BOUNDING FACE i FOR $1 \le i \le f$,
SO $a_i \ge 3$ FOR EACH i. SUPPOSE $a_i > 3$ FOR SOME i;
THEN $3f < a_1 + \cdots + a_f \le 2e$ SINCE EVERY EDGE BOUNDING A FACE
BOUNDS EXACTLY 2 FACES. SINCE $n - e + f = 2$, $3n - 3e + 3f = 6$
SO $6 = 3n - 3e + 3f < 3n - 3e + 2e = 3n - e \Rightarrow e < 3n - 6$,
WHICH GIVES A CONTRADICTION. THEREFORE $a_i = 3$ FOR EACH i.

(27) SUPPOSE INSTEAD THAT G HAS AT MOST ONE VERTEX v WITH $\deg(v) \le 5$.
THEN $d_1 + \cdots + d_n = 2e$ WHERE, SAY, $d_i \ge 6$ FOR $2 \le i \le n$;
SO $2e = d_1 + (d_2 + \cdots + d_n) \ge d_1 + 6(n-1) > 6(n-1) \Rightarrow e > 3n - 3$.
IF G IS A CONNECTED PLANAR GRAPH, THIS GIVES A CONTRADICTION
SINCE $e \le 3n - 6$; SO G HAS AT LEAST 2 VERTICES WITH $\deg(v) \le 5$.
(IF G IS NOT CONNECTED, WE CAN APPLY THIS ARGUMENT TO ANY
CONNECTED COMPONENT OF G.)

GOLD – (1) A) 1) WHEN $e = 0$, $G = N_n$ SO G HAS $n = n - 0$ CONNECTED COMPONENTS.

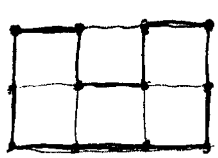
2) ASSUME THAT THIS STATEMENT IS TRUE FOR A GIVEN VALUE OF $e \ge 0$,
AND LET G HAVE $e+1$ EDGES.
IF uv IS ANY EDGE OF G, THEN $H = G - uv$ HAS e EDGES;
SO H HAS k CONNECTED COMPONENTS WHERE $k \ge n - e$ (BY THE INDUCTION HYPOTHESIS)

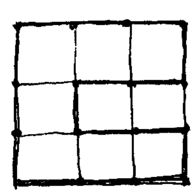
a) IF u AND v ARE IN THE SAME COMPONENT OF H,
THEN G ALSO HAS k COMPONENTS.

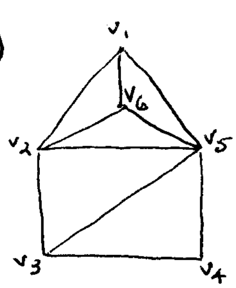
b) IF u AND v ARE IN DIFFERENT COMPONENTS OF H,
THEN G HAS $k-1$ COMPONENTS.

SINCE $k \ge k - 1 \ge n - e - 1 = n - (e+1)$,
G HAS AT LEAST $n - (e+1)$ COMPONENTS IN EITHER CASE;
SO THE STATEMENT IS TRUE FOR $e+1$.

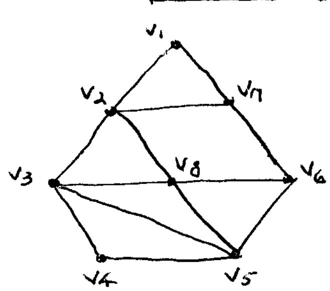
B) IF G IS CONNECTED, THEN $1 \ge n - e$ BY PART A); SO $e \ge n - 1$.

(2) A)  HAS A HAMILTON CYCLE

B)  HAS A HAMILTON CYCLE

(3) 

$v_1 \to C_1$
$v_2 \to C_3$
$v_3 \to C_1$
$v_4 \to C_2$
$v_5 \to C_3$
$v_6 \to C_4$



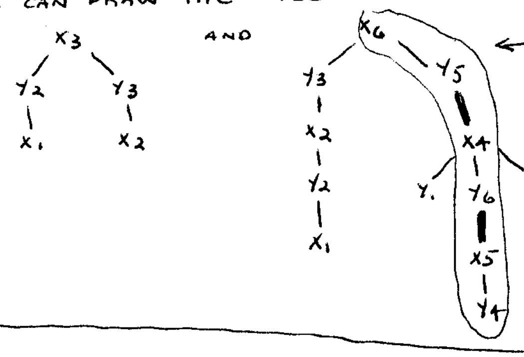
$v_1 \to C_1$
$v_2 \to C_2$
$v_3 \to C_1$
$v_4 \to C_2$
$v_5 \to C_3$
$v_6 \to C_1$
$v_7 \to C_3$
$v_8 \to C_4$

MATH 145 **SOLUTIONS – GOLD AND ORANGE SHEETS**

G.S. – 4 a) IF $S = \{x_1, x_2, x_3\}$, THEN $N(S) = \{y_2, y_3\}$; SO $|S| > |N(S)|$ AND SO THERE IS NO MATCHING OF X INTO Y.

b) $M = \{x_1y_2, x_2y_3, x_4y_7, x_5y_6, x_6y_5\}$ IS A MATCHING WITH 5 EDGES, SO BY PART A) IT IS A MAXIMUM MATCHING.

REMARK IN b), IF WE START INSTEAD WITH $M = \{x_1y_2, x_2y_3, x_4y_5, x_5y_6\}$, WE CAN DRAW THE TREES FOR THE UNSATURATED VERTICES x_3 AND x_6 TO GET



← THIS GIVES AN M-AUGMENTING PATH, SO REPLACING x_4y_5 AND x_5y_6 BY x_6y_5, x_4y_6, AND x_5y_4 GIVES A LARGER MATCHING $M' = \{x_1y_2, x_2y_3, x_4y_6, x_5y_4, x_6y_5\}$

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5 LET G BE THE BIPARTITE GRAPH WITH VERTICES $A = \{r_1, \dots, r_{13}\}$ AND $B = \{p_1, \dots, p_{13}\}$ CORRESPONDING TO THE 13 RANKS AND 13 PILES, WITH AN EDGE BETWEEN p_i AND r_j IF PILE i CONTAINS A CARD OF RANK j.

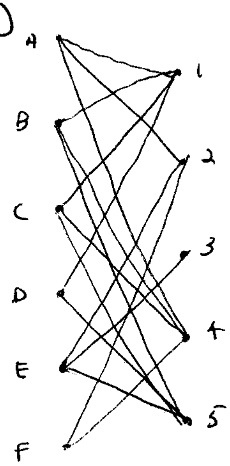
IF S IS ANY SUBSET OF A WITH $|S| = k$, THEN THERE ARE $4k$ CARDS OF THE RANKS IN S. IF THESE CARDS APPEAR IN ℓ PILES, THEN $4k \le 4\ell$ (SINCE THESE PILES CONTAIN 4ℓ CARDS) SO $k \le \ell$.

THEREFORE $|S| = k \le \ell = |N(S)|$,

SO G HAS A MATCHING THAT SATURATES A BY HALL'S TH.

REMARK SINCE $|A| = |B|$, THE MATCHING ALSO SATURATES B AND IS A PERFECT MATCHING.

O.S. – 1



IF $X = \{A, B, C, D, E, F\}$ AND $Y = \{1, 2, 3, 4, 5\}$, A MAXIMUM MATCHING M CAN HAVE AT MOST 5 EDGES SINCE $|Y| = 5$;

SO $M = \{A4, B5, C1, E3, F2\}$ IS A MAXIMUM MATCHING.

MATH 145 SOLUTIONS – ORANGE SHEET

② A) THE 6 VERTICES CORRESPONDING TO ALL THE WOMEN EXCEPT HELEN HAVE ONLY 5 NEIGHBORS A, B, C, E, AND F; SO BY HALL'S TH. THERE IS NO MATCHING WHICH SATURATES THE VERTICES FOR THE WOMEN AND THEREFORE IT IS NOT THE CASE THAT EVERY WOMAN CAN FILL A POSITION.

B) BY PART A), ANY VALID MATCHING FOR 6 OF THE WOMEN, SUCH AS $M = \{aF, bB, cE, hH, mA, sC\}$, IS A MAXIMUM MATCHING.

③ LET $A_1, \ldots, A_n$ BE SUBSETS OF A SET S.
IT IS POSSIBLE TO SELECT DISTINCT ELEMENTS $s_1, \ldots, s_n$ OF S WITH $s_i \in A_i$ FOR EACH i IFF FOR EVERY SUBSET I OF $\{1, \ldots, n\}$,

$$|I| \le \left| \bigcup_{i \in I} A_i \right|.$$

PF LET G BE A BIPARTITE GRAPH WITH VERTICES $A = \{A_1, \ldots, A_n\}$ AND $S = \{s_1, \ldots, s_m\}$, WITH AN EDGE BETWEEN A_i AND s_j IFF $s_j \in A_i$.

IT IS POSSIBLE TO SELECT DISTINCT ELEMENTS $s_1, \ldots, s_n$ OF S WITH $s_i \in A_i$ FOR EACH i IFF

G HAS A MATCHING WHICH SATURATES A IFF

$|T| \le |N(T)|$ FOR EVERY SUBSET T OF A BY HALL'S TH.

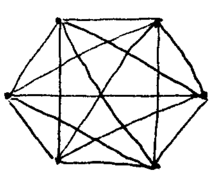
IF $T = \{A_i : i \in I\}$ WHERE $I \subseteq \{1, \ldots, n\}$,

THEN $|T| = |I|$ AND $N(T) = \bigcup_{i \in I} A_i$; SO $|T| \le |N(T)|$ IFF $|I| \le \left| \bigcup_{i \in I} A_i \right|$.

THEREFORE G HAS A MATCHING WHICH SATURATES A IFF

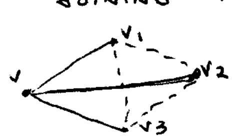
$$|I| \le \left| \bigcup_{i \in I} A_i \right| \text{ FOR EVERY SUBSET } I \text{ OF } \{1, \ldots, n\}.$$

④



SELECT ANY VERTEX v. SINCE IT IS INCIDENT TO 5 EDGES, AT LEAST 3 OF THESE EDGES MUST HAVE THE SAME COLOR (BY THE STRONG FORM OF THE PH PRINCIPLE, SINCE WE'RE PUTTING 5 OBJECTS INTO 2 BOXES.)

WE CAN ASSUME THAT THERE ARE 3 BLUE EDGES, SAY, SO LET THEIR OTHER VERTICES BE v_1, v_2, v_3. IF AN EDGE JOINING v_i AND v_j IS BLUE, THEN $v v_i v_j$ GIVES A BLUE TRIANGLE; AND IF THE EDGES JOINING v_1, v_2, v_3 ARE ALL GREEN, THEN WE HAVE A GREEN TRIANGLE.



REMARK SEE PP. 77-79 FOR MORE INFORMATION ABOUT THIS.