CH. 12 - (2)

Let \( a_i \) be the number of edges bounding face \( i \). For \( 1 \leq i \leq f \),

\[ 2a_i = 3 \] for each \( i \).

Suppose \( a_i \geq 3 \) for some \( i \); then \( 2f \leq 3e \), since every edge bounding a face

bounds exactly 2 faces. Since \( 3n - 3e + 3f = 6 \)

so \( 6 = 3n - 3e + 3f < 3n - 3e + 2e = 3n - e \Rightarrow e > 3n - 6 \),

which gives a contradiction. Therefore \( a_i = 3 \) for each \( i \).

(2) Suppose instead that \( G \) has at most one vertex \( v \) with \( \text{deg}(v) \leq 5 \).

Then \( 2e = d_1 + d_2 + \cdots + d_n \) where, say, \( d_i \geq 6 \) for \( 2 \leq i \leq n \);

so \( 2e = d_1 + (d_2 + \cdots + d_n) \geq d_1 + 6(n-1) > 6(n-1) \Rightarrow e > 3n - 6 \).

If \( G \) is a connected planar graph, this gives a contradiction

since \( e \leq 3n - 6 \); so \( G \) has at least 2 vertices with \( \text{deg}(v) \leq 5 \).

(If \( G \) is not connected, we can apply this argument to any

connected component of \( G \).)

GOLD - (1)

A) When \( e = 0 \), \( G = \emptyset \) so \( G \) has \( n = n - 0 \) connected components.

1) Assume that this statement is true for a given value of \( e \geq 0 \),

and let \( G \) have \( e + 1 \) edges.

If \( uv \) is any edge of \( G \), then \( H = G - uv \) has \( e \) edges;

so \( H \) has \( k \) connected components where \( k > n - e \) (by the induction hypothesis).

a) If \( u \) and \( v \) are in the same component of \( H \),

then \( G \) also has \( k \) components.

b) If \( u \) and \( v \) are in different components of \( H \),

then \( G \) has \( k - 1 \) components,

since \( k > k - 1 \) \( \geq n - e - 1 = n - (e + 1) \).

\( G \) has at least \( n - (e + 1) \) components in either case;

so the statement is true for \( e + 1 \).

B) if \( G \) is connected, then \( 1 \geq n - e \) by part A); so \( e \geq n - 1 \),

A)

Has a Hamilton cycle

B)

Has a Hamilton cycle

3)

V1 \( \rightarrow \) C1
V2 \( \rightarrow \) C2
V3 \( \rightarrow \) C1
V4 \( \rightarrow \) C2
V5 \( \rightarrow \) C3
V6 \( \rightarrow \) C4
G.3. (4) a) If $S = \{x_1, x_2, x_3, x_4\}$, then $N(S) = \{y_2, y_3, y_4\}$, so $|S| > |N(S)|$ and so there is no matching of $X$ into $Y$.

b) $M = \{x_1y_2, x_2y_3, x_4y_4, x_5y_6, x_6y_5\}$ is a matching with 5 edges, so by Part A) it is a maximum matching.

**Remark** in b), if we start instead with $M = \{x_1y_2, x_2y_3, x_4y_5, x_5y_6\}$, we can draw the trees for the unsaturated vertices $x_3$ and $x_6$ to get $X_3$, $X_6$. This gives an $M$-augmenting path, so replacing $x_4y_5$ and $x_5y_6$ by $x_6y_5$, $x_4y_6$, and $x_5y_4$ gives a larger matching $M' = \{x_1y_2, x_2y_3, x_4y_6, x_5y_4, x_6y_5\}$.

5. Let $G$ be the bipartite graph with vertices $A = \{x_1, \ldots, x_{13}\}$ and $B = \{y_1, \ldots, y_{13}\}$ corresponding to the 13 ranks and 13 piles, with an edge between $p_i$ and $y_j$ if pile $i$ contains a card of rank $j$.

If $S$ is any subset of $A$ with $|S| = k$, then there are $4k$ cards of the ranks in $S$. If these cards appear in $\ell$ piles, then $4k \leq 4\ell$ (since these piles contain $4k$ cards), so $k \leq \ell$.

Therefore $|S| = k \leq \ell = |N(S)|$, so $G$ has a matching that saturates $A$ by Hall’s Th.

**Remark** since $|A| = |B|$, the matching also saturates $B$ and is a perfect matching.

0.3. (4) If $X = \{A, B, C, D, E, F\}$ and $Y = \{1, 2, 3, 4, 5\}$, a maximum matching $M$ can have at most 5 edges since $|Y| = 5$.

So $M = \{A4, B5, C1, E3, F2\}$ is a maximum matching.
A) The 6 vertices corresponding to all the women except Helen have only 5 neighbors A, B, C, E, and F; so by Hall's Th. there is no matching which saturates the vertices for the women and therefore it is not the case that every woman can fill a position.

B) By part A), any valid matching for 6 of the women, such as
\[ M = \{ aF, bB, cE, hH, mA, sC \} \], is a maximum matching.

3) Let \( A_1, \ldots, A_n \) be subsets of a set \( S \).

It is possible to select distinct elements \( s_1, \ldots, s_n \) of \( S \) with \( s_i \in A_i \) for each \( i \) iff for every subset \( I \) of \( \{1, \ldots, n\} \),
\[ |I| \leq |U_A| \] 

Let \( G \) be a bipartite graph with vertices \( A = \{ A_1, \ldots, A_n \} \) and \( S = \{ s_1, \ldots, s_m \} \), with an edge between \( A_i \) and \( S_j \) iff \( S_j \in A_i \).

It is possible to select distinct elements \( s_1, \ldots, s_n \) of \( S \) with \( s_i \in A_i \) for each \( i \) iff \( G \) has a matching which saturates \( A \) iff \( |T| \leq |N(T)| \) for every subset \( T \) of \( A \) by Hall's Th., if \( T = \{ A_i \mid i \in I \} \) where \( I \subseteq \{1, \ldots, n\} \),
then \( |T| = |I| \) and \( N(T) = U_{i \in I} A_i \); so \( |T| \leq |N(T)| \) iff \( |I| \leq |U_{i \in I} A_i| \).

Therefore, \( G \) has a matching which saturates \( A \) iff
\[ |I| \leq |U_{i \in I} A_i| \] 
for every subset \( I \) of \( \{1, \ldots, n\} \).

A) Select any vertex \( v \), since it is incident to 5 edges, at least 3 of these edges must have the same color (by the strong form of the Pigeonhole Principle, since we're putting 5 objects into 2 boxes.)

We can assume that there are 3 blue edges, say, so let their other vertices be \( v_1, v_2, v_3 \). If an edge joining \( v_1 \) and \( v_1' \) is blue, then \( v_1, v_1', v_1 \) gives a blue triangle; and if the edges joining \( v_1, v_2, v_3 \) are all green, then we have a green triangle.

Remark: see pp. 77-79 for more information about this.