MATH 145

# CH. 2 - SOLUTIONS

(22) WE CONSIDER CASES, ACCORDING TO HOW MANY PEOPLE TIE:

a) NO TIES: $4! = \underline{24}$ WAYS

b) 2 PEOPLE TIE: $\binom{4}{2} \cdot 3! = \underline{36}$ WAYS

c) 2 PAIRS TIE: $3 \cdot 2! = \underline{6}$ WAYS (SINCE ARE 3 WAYS TO GET 2 PAIRS)

d) 3 PEOPLE TIE: $\binom{4}{3} \cdot 2! = \underline{8}$ WAYS

e) ALL 4 TIE: $\underline{1}$ WAY

TOTAL: $24 + 36 + 6 + 8 + 1 = \boxed{75}$

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ALTERNATE SOLUTION

LET $a_n$ BE THE NUMBER OF POSSIBILITIES IF THERE ARE $n$ RUNNERS.

IF $k$ PEOPLE FINISH FIRST, THEN THERE ARE $a_{n-k}$ POSSIBILITIES FOR THE OTHER RUNNERS,

SO $\underline{a_n = \sum_{k=1}^{n} \binom{n}{k} a_{n-k}}$, TAKING $a_0 = 1$.

THEN $\underline{a_1} = 1$, $\underline{a_2} = \binom{2}{1} a_1 + \binom{2}{2} a_0 = 3$, $\underline{a_3} = \binom{3}{1} a_2 + \binom{3}{2} a_1 + \binom{3}{3} a_0 = 3 \cdot 3 + 3 \cdot 1 + 1 = 13$,

$\underline{a_4} = \binom{4}{1} a_3 + \binom{4}{2} a_2 + \binom{4}{3} a_1 + \binom{4}{4} a_0 = 4 \cdot 13 + 6 \cdot 3 + 4 \cdot 1 + 1 = \boxed{75}$.

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(37) a) 12 DOTS, 5 DIVIDERS: $\boxed{\binom{17}{5}}$

b) FIRST TAKE ONE OF EACH KIND OF PASTRY.

6 DOTS, 5 DIVIDERS: $\boxed{\binom{11}{5}}$

ALTERNATE SOLUTION

$\cdot_1 \ \cdot_2 \ \cdot_3 \ \cdots \ \cdot_{12}$ LINE UP 12 DOTS; THERE ARE 11 INNER GAPS, AND WE MUST CHOOSE 5 OF THEM IN WHICH TO PUT THE DIVIDERS: $\boxed{\binom{11}{5}}$

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(38) $\underline{x_1 + x_2 + x_3 + x_4 = 30}$, $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, $x_4 \geq 8$

LET $y_1 = x_1 - 2$, $y_2 = x_2$, $y_3 = x_3 + 5$, $y_4 = x_4 - 8$.

THEN $(y_1 + 2) + y_2 + (y_3 - 5) + (y_4 + 8) = 30$,

SO $\underline{y_1 + y_2 + y_3 + y_4 = 25}$ WITH $y_i \geq 0$ FOR EACH $i$.

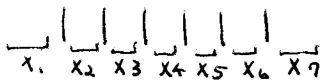
3 DIVIDERS, 25 DOTS: $\boxed{\binom{28}{3}}$

(39) a) $\boxed{\binom{20}{6}}$

b) LINE UP 14 STICKS, REPRESENTING THE ONES NOT CHOSEN. $\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}\ \cdots\ \underset{14}{|}$

THIS GIVES 15 GAPS, AND WE MUST CHOOSE 6 OF THEM FOR THE CHOSEN STICKS: $\boxed{\binom{15}{6}}$

(39) b) ALTERNATE SOLUTION




LET $x_i$ BE THE NUMBER OF STICKS IN GAP $i$, FOR EACH OF THE 7 GAPS CREATED BY THE STICKS CHOSEN.

THEN $x_1 + \cdots + x_7 = 14$ WITH $x_i \geq 1$ FOR $2 \leq i \leq 6$ AND $x_1, x_7 \geq 0$.

LET $y_1 = x_1$, $y_7 = x_7$, AND $y_i = x_i - 1$ FOR $2 \leq i \leq 6$.

THEN $y_1 + \cdots + y_7 = 9$ WITH $y_i \geq 0$ FOR EACH $i$,

SO THERE ARE $\boxed{\binom{15}{6}}$ SOLUTIONS (9 DOTS, 6 DIVIDERS)

c) AS IN b), $x_1 + \cdots + x_7 = 14$ WITH $x_i \geq 2$ FOR $2 \leq i \leq 6$ AND $x_1, x_7 \geq 0$.

LET $y_1 = x_1$, $y_7 = x_7$, AND $y_i = x_i - 2$ FOR $2 \leq i \leq 6$.

THEN $y_1 + \cdots + y_7 = 4$ WITH $y_i \geq 0$ FOR EACH $i$,

SO THERE ARE $\boxed{\binom{10}{6}}$ SOLUTIONS (4 DOTS, 6 DIVIDERS)

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ALTERNATE SOLUTION

COLOR THE 6 CHOSEN STICKS RED AND THE 14 STICKS NOT CHOSEN BLUE, AND REMOVE 10 BLUE STICKS (TO ACT AS "BLOCKERS").
THERE ARE 6 RED STICKS AND 4 BLUE STICKS LEFT, AND THEY CAN BE ARRANGED IN $\boxed{\binom{10}{4}}$ WAYS. (THEN INSERT 2 "BLOCKERS" IN EACH GAP BETWEEN THE RED STICKS.)

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(42) 1) GIVE AWAY THE LEMON DRINK: 4 CHOICES.
2) GIVE AWAY THE LIME DRINK: 3 CHOICES.
NOW GIVE AN ORANGE DRINK TO THE OTHER 2 STUDENTS.
3) GIVE THE 8 REMAINING ORANGE DRINKS TO THE 4 STUDENTS:
$\binom{11}{3}$ CHOICES (8 DOTS, 3 DIVIDERS)

ANSWER: $\boxed{4 \cdot 3 \cdot \binom{11}{3}}$