 CH. 5 – SOLUTIONS

① $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

PF $\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{(n-1)!(k)}{k!(n-k)!}$

$= \frac{(n-1)!\,[n-k+k]}{k!(n-k)!} = \frac{(n-1)!(n)}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$

⑤ $(2x-y)^7 = \sum_{k=0}^{7} \binom{7}{k} (2x)^{7-k} (-y)^k$

$= (2x)^7 + 7(2x)^6(-y) + 21(2x)^5(-y)^2 + 35(2x)^4(-y)^3 + 35(2x)^3(-y)^4 + 21(2x)^2(-y)^5 + 7(2x)(-y)^6 + (-y)^7$

$= 128x^7 - 448x^6y + 672x^5y^2 - 560x^4y^3 + 280x^3y^4 - 84x^2y^5 + 14xy^6 - y^7$

⑦ $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, so A) LETTING $x=2$ GIVES $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$.

B) LETTING $x=r$ GIVES $(1+r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k$

⑧ $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, SO LETTING $x=3$ AND $y=-1$ GIVES

$2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}.$

⑨ $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, SO $\sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^k = \sum_{k=0}^{n} \binom{n}{k} (-10)^k = (1+(-10))^n = (-9)^n$

⑩ SUPPOSE WE WANT TO SELECT A TEAM OF k PEOPLE FROM A GROUP OF n PEOPLE, AND A CAPTAIN FOR THE TEAM.

1) IF WE PICK THE TEAM AND THEN THE CAPTAIN, THERE ARE $\binom{n}{k} \cdot k$ POSSIBILITIES.

2) IF WE PICK THE CAPTAIN AND THEN THE REMAINING $k-1$ PEOPLE, THERE ARE $n \cdot \binom{n-1}{k-1}$ CHOICES.

THEREFORE $k\binom{n}{k} = n\binom{n-1}{k-1}$.

⑮ $\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots + (-1)^{n-1} n \binom{n}{n}$

$= n\binom{n-1}{0} - n\binom{n-1}{1} + n\binom{n-1}{2} - \cdots + n(-1)^{n-1}\binom{n-1}{n-1}$ (USING #10)

$= n\left[\binom{n-1}{0} - \binom{n-1}{1} + \binom{n-1}{2} - \cdots + (-1)^{n-1}\binom{n-1}{n-1}\right] = n \cdot 0 = 0$ FOR $n > 1$

(USING 5.4).

(OR USE THE BINOMIAL TH., AS ON THE NEXT PAGE)

(15) $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, SO DIFFERENTIATING GIVES $n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1}$

LETTING $x=-1$ GIVES $\sum_{k=1}^{n} (-1)^{k-1} k \binom{n}{k} = 0$.

(21) $\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}$ BOTH SIDES ARE 1 IF $k=0$
BOTH SIDES ARE 0 IF $k<0$

IF $k>0$, $\binom{-r}{k} = \frac{(-r)(-r-1)(-r-2)\cdots(-r-k+1)}{k!} = \frac{(-1)^k (r)(r+1)\cdots(r+k-1)}{k!}$

$= (-1)^k \frac{(r+k-1)\cdots(r+1)(r)}{k!} = (-1)^k \binom{r+k-1}{k}$.

(22) $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$

a) WE CAN ASSUME $m \geq 0$, $k \geq 0$, AND $m-k \geq 0$, SINCE OTHERWISE BOTH SIDES ARE 0,

b) IF $k=0$, BOTH SIDES ARE $\binom{r}{m}$; SO WE CAN ASSUME $m \geq k > 0$, THEN

$\binom{r}{m}\binom{m}{k} = \frac{r(r-1)\cdots(r-m+1)}{m!} \cdot \frac{m!}{k!\,(m-k)!} = \frac{r(r-1)\cdots(r-m+1)}{k!\,(m-k)!}$ AND

$\binom{r}{k}\binom{r-k}{m-k} = \frac{r(r-1)\cdots(r-k+1)}{k!} \cdot \frac{(r-k)(r-k-1)\cdots(r-m+1)}{(m-k)!} = \frac{r(r-1)\cdots(r-m+1)}{k!\,(m-k)!}$,

SO $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$.

(28) $\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$.

SUPPOSE WE WANT TO SELECT A COMMITTEE OF n PEOPLE FROM A GROUP OF n MEN AND n WOMEN, INCLUDING A WOMAN CHAIR OF THE COMMITTEE,

1) IF WE SELECT THE WOMAN CHAIR OF THE COMMITTEE FIRST AND THEN THE REMAINING $n-1$ MEMBERS, WE HAVE $n\binom{2n-1}{n-1}$ CHOICES.

2) IF WE SELECT k WOMEN FOR THE COMMITTEE FIRST (WITH $k \geq 1$), THERE ARE $\binom{n}{k}$ WAYS TO DO THIS, THEN WE HAVE k CHOICES FOR THE CHAIR OF THE COMMITTEE, AND $\binom{n}{n-k} = \binom{n}{k}$ WAYS TO CHOOSE THE $n-k$ MEN ON THE COMMITTEE.
BY THE ADDITION PRINCIPLE, THIS GIVES $\sum_{k=1}^{n} k \binom{n}{k}\binom{n}{n-k} = \sum_{n=1}^{n} k \binom{n}{k}^2$ CHOICES.

THEREFORE $\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$.