CH. 6 - SOLUTIONS

2. Let \( S = \{1, 2, \ldots, 10000\} \), \( A_1 \) be the multiples of 4 in \( S \), \( A_2 \) the multiples of 6 in \( S \), \( A_3 \) the multiples of 7 in \( S \), and \( A_4 \) the multiples of 10 in \( S \).

\[
|A_1 \cap A_2 | = 101 - \sum_{i<j} |A_i \cap A_j| - \sum_{i \neq j} |A_i \cap A_j| + 14 \cdot n \cdot A_4 | = 10000 - 2500 - 1666 - 1428 - 1000 + 833 + 357 + 500 + 488 + 571 + 14
\]
\[
\left(\begin{array}{c} 10000 \end{array}\right) \left(\begin{array}{c} 2500 \end{array}\right) \left(\begin{array}{c} 1666 \end{array}\right) \left(\begin{array}{c} 1428 \end{array}\right) \left(\begin{array}{c} 1000 \end{array}\right) \left(\begin{array}{c} 833 \end{array}\right) \left(\begin{array}{c} 357 \end{array}\right) \left(\begin{array}{c} 500 \end{array}\right) \left(\begin{array}{c} 488 \end{array}\right) \left(\begin{array}{c} 571 \end{array}\right) \left(\begin{array}{c} 14 \end{array}\right)
\right)
\]
\[
- 1111 - 166 - 71 - 47 + 23 = 5667
\]

3. Let \( S = \{1, 2, \ldots, 10000\} \), \( A \) the perfect squares in \( S \), and \( B \) as the perfect cubes in \( S \).

\[
|A \cap B| = 131 - |A| - |B| + (|A \cap B| = 10000 - 100 - 21 + 3
\]

(4546 that \( 10000 = 100 \), \( \sqrt{10000} \approx 21.5 \), \( \sqrt{10000} \approx 46 \))

4. Let \( T \) be the 10-combinations of \( \{10, 0, 10, b, 10, c, 10, d, 0\} \), and

Let \( A_1 \) be the elements of \( T \) with at least 5 b's,
Let \( A_2 \) be the elements of \( T \) with at least 6 c's, and
Let \( A_3 \) be the elements of \( T \) with at least 8 d's.

Then

\[
|A_1 \cap A_2 \cap A_3| = 111 - \sum_{i<j} |A_i \cap A_j| - \sum_{i \neq j} |A_i \cap A_j| - \sum_{i \neq j} |A_i \cap A_j| - \sum_{i \neq j} |A_i \cap A_j|
\]

\[
\left(\begin{array}{c} 111 \end{array}\right) \left(\begin{array}{c} 5 \end{array}\right) \left(\begin{array}{c} 6 \end{array}\right) \left(\begin{array}{c} 8 \end{array}\right)
\right)
\]

(185 since all double intersections are empty)

5. Let \( T \) be the set of options if there were 12 of each type available, and

Let \( A_1 \) be the options with at least 7 chocolate, \( A_2 \) the options with at least 7 cinnamon, and \( A_3 \) the options with at least 4 plain.

Then

\[
|A_1 \cap A_2 \cap A_3| = 111 - \sum_{i<j} |A_i \cap A_j| - \sum_{i \neq j} |A_i \cap A_j| - \sum_{i \neq j} |A_i \cap A_j|
\]

\[
\left(\begin{array}{c} 111 \end{array}\right) \left(\begin{array}{c} 7 \end{array}\right) \left(\begin{array}{c} 7 \end{array}\right) \left(\begin{array}{c} 4 \end{array}\right)
\right)
\]

(10)

OR CONSIDER CASES, BASED ON HOW MANY PLAIN ARE CHOSEN:

a) 0 plain: \( 1 \) option (6 ch., 6 ci.)

b) 1 plain: \( 3 \) options (6 ch., 5 ci. or 5 ch., 6 ci.)

c) 2 plain: \( 3 \) options (6 ch., 4 ci. or 5 ch., 5 ci. or 4 ch., 6 ci.)

d) 3 plain: \( 4 \) options (6 ch., 3 ci. or 5 ch., 4 ci. or 4 ch., 5 ci. or 3 ch., 6 ci.)

TOTAL: \( 1 + 2 + 3 + 4 = 10 \) options
8. \( x_1 + x_2 + x_3 + x_4 = 20, \ \ 1 \leq x_1 \leq 6, \ 0 \leq x_2 \leq 7, \ 4 \leq x_3 \leq 8, \ 2 \leq x_4 \leq 6 \)

Let \( y_1 = x_1 - 1, \ y_2 = x_2, \ y_3 = x_3 - 4, \ y_4 = x_4 - 2 \) to get

\[ y_1 + y_2 + y_3 + y_4 = 13 \quad \text{with} \quad 0 \leq y_1 \leq 5, \ 0 \leq y_2 \leq 7, \ 0 \leq y_3 \leq 4, \ 0 \leq y_4 \leq 4 \]

Let \( S \) be the set of non-negative integral solutions, \( A \) the elements of \( S \) with \( y_1 \geq 6 \), \( B \) the elements of \( S \) with \( y_2 \geq 8 \), \( C \) the elements of \( S \) with \( y_3 \geq 5 \), and \( D \) the elements of \( S \) with \( y_4 \geq 5 \).

Then

\[
|A \cap B \cap C| = 15! - \sum_{i \leq j} |A_i \cap A_j| - \sum_{i \leq j \leq k} |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|
\]

\[
= \left( \binom{15}{3} - \binom{10}{3} - \binom{8}{3} - \binom{11}{3} - \binom{3}{3} + \binom{5}{3} + \binom{5}{3} + \binom{3}{3} + \binom{6}{3} \right) = 96
\]

9. Let \( T \) be the set of permutations of \( S = \{3, a, 4, b, 4, c\} \), and let \( A \) be the permutations with the \( a \)'s consecutive, \( B \) be the permutations with the \( b \)'s consecutive, and \( C \) be the permutations with the \( c \)'s consecutive.

Then

\[
|A \cap B \cap C| = |T| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|
\]

\[
= \frac{9!}{3!3!3!} - \frac{7!}{4!2!3!} - \frac{6!}{3!2!3!} - \frac{8!}{3!4!2!} + \frac{4!}{4!2!3!} + \frac{6!}{4!2!3!} + \frac{5!}{3!3!3!} - 3!
\]

\[
= 871
\]

10. Let \( S \) be the set of selections of 10 stops for the 10 people without restrictions, and let \( A_i \) be the selections of 10 stops where no one exits at stop \( i \), \( 1 \leq i \leq 6 \).

Then

\[
|A_1 \cap \ldots \cap A_6| = 15! - \sum_{i \leq j} |A_i \cap A_j| - \sum_{i \leq j \leq k} |A_i \cap A_j \cap A_k| + \ldots + |A_1 \cap A_2 \cap \ldots \cap A_6|
\]

\[
= 6^{10} - \binom{6}{1} 5^{10} + \binom{6}{2} 4^{10} - \binom{6}{3} 3^{10} + \binom{6}{4} 2^{10} - \binom{6}{5} 1^{10}
\]

\[
= 16,435,440
\]