MATH 145

CH. 6 – SOLUTIONS

(11) LET S BE THE SET OF PERMUTATIONS OF $\{1, \ldots, 8\}$, AND

LET A_i BE THE PERMUTATIONS IN S WHICH LEAVE $2i$ IN ITS NATURAL POSITION, $1 \le i \le 4$.

THEN $|\overline{A_1} \cap \cdots \cap \overline{A_4}| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + |A_1 \cap \cdots \cap A_4|$

$$= \boxed{8! - \binom{4}{1}7! + \binom{4}{2}6! - \binom{4}{3}5! + 4!} = \boxed{24{,}024}$$

(12) 1) CHOOSE THE 4 INTEGERS LEFT IN THEIR NATURAL POSITIONS: $\binom{8}{4}$ WAYS

2) ARRANGE THE OTHER 4 INTEGERS SO THAT THEY ARE NOT LEFT FIXED: D_4 WAYS

ANSWER: $\binom{8}{4} \cdot D_4 = \boxed{\binom{8}{4} \cdot 9} = \boxed{630}$

(13) LET S BE THE SET OF PERMUTATIONS OF $\{1, \ldots, 9\}$, AND

LET A_i BE THE PERMUTATIONS IN S WHICH LEAVE $2i-1$ FIXED, FOR $1 \le i \le 5$.

THEN $|A_1 \cup \cdots \cup A_5| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + |A_1 \cap \cdots \cap A_5|$

$$= \boxed{\binom{5}{1}8! - \binom{5}{2}7! + \binom{5}{3}6! - \binom{5}{4}5! + 4!} = \boxed{157{,}824}$$

(15) a) $D_7 = \boxed{1854} \longleftarrow D_7 = 7D_6 - 1 = 7(265) - 1$

b) $7! - D_7 = 5040 - 1854 = \boxed{3186}$

c) $|A| = |S| - |A^c| = 7! - D_7 - \binom{7}{1}D_6$

$= 5040 - 1854 - 1855 = \boxed{1331}$,

SINCE THERE ARE D_7 WAYS THAT NO MAN RECEIVES HIS OWN HAT

AND $\binom{7}{1}D_6$ WAYS THAT EXACTLY ONE MAN RECEIVES HIS OWN HAT.

MATH 145 WHITE SHEET – SOLUTIONS

(1) LET S BE THE SET OF ALL 13-CARD HANDS, AND
LET A_i BE THE SET OF 13-CARD HANDS WITH NO CARDS FROM SUIT i, FOR $1 \le i \le 4$
(WITH THE SUITS ORDERED ALPHABETICALLY, SAY).

THEN $$|\bar{A}_1 \cap \cdots \cap \bar{A}_4| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + |A_1 \cap \cdots \cap A_4|$$

$$= \boxed{\binom{52}{13} - \binom{4}{1}\binom{39}{13} + \binom{4}{2}\binom{26}{13} - \binom{4}{3}\binom{13}{13}} = \underline{602,586,261,420}$$

(2) LET S BE THE SET OF SELECTIONS IF 15 OF EACH TYPE WERE AVAILABLE,
LET A_1 BE THE SELECTIONS WITH AT LEAST 7 BLUEBERRY,
LET A_2 BE THE SELECTIONS WITH AT LEAST 6 CHEESE, AND
LET A_3 BE THE SELECTIONS WITH AT LEAST 4 CINNAMON-RAISIN.

THEN $$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3|$$

$$= \boxed{\underbrace{\binom{18}{3}}_{|S|} - \underbrace{\binom{11}{3}}_{|A_1|} - \underbrace{\binom{12}{3}}_{|A_2|} - \underbrace{\binom{14}{3}}_{|A_3|} + \underbrace{\binom{5}{3}}_{|A_1 \cap A_2|} + \underbrace{\binom{7}{3}}_{|A_1 \cap A_3|} + \underbrace{\binom{8}{3}}_{|A_2 \cap A_3|}} = \boxed{168}$$

(4)

1	2	3
4	5	6
7	8	9

LET S BE THE SET OF ALL COLORINGS OF THE SQUARES, AND
LET A_i BE THE SET OF COLORINGS FOR WHICH THE iTH 2×2 SQUARE IS ALL GREEN (WHERE THE 2×2 SQUARES ARE ORDERED IN INCREASING ORDER OF THEIR UPPER LEFT CORNERS: 1, 2, 4, 5).

THEN $$|\bar{A}_1 \cap \cdots \cap \bar{A}_4| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + |A_1 \cap \cdots \cap A_4|$$

$$= \boxed{2^9 - \binom{4}{1}\cdot 2^5 + \left[4 \cdot 2^3 + 2 \cdot 2^2\right] - \binom{4}{3}\cdot 2^1 + 1} = \boxed{417}$$

(SINCE $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4| = 2^3$

AND $|A_1 \cap A_4| = |A_2 \cap A_3| = 2^2$)

((3) WAS WORKED IN CLASS.)

WHITE SHEET – SOLUTIONS

(5) A) LET $A = \{1, \ldots, 11\}$ AND $B = \{1, \ldots, 5\}$, AND

LET S BE THE SET OF FUNCTIONS $f : A \to B$.

LET E_i BE THE FUNCTIONS IN S WHICH DO NOT TAKE THE VALUE i, FOR $1 \le i \le 5$.

THEN $|\overline{E_1} \cap \cdots \cap \overline{E_5}| = |S| - \sum_i |E_i| + \sum_{i<j} |E_i \cap E_j| - \cdots - |E_1 \cap \cdots \cap E_5|$

$$= \boxed{5^{11} - \binom{5}{1}4^{11} + \binom{5}{2}3^{11} - \binom{5}{3}2^{11} + \binom{5}{4}1^{11}} = \boxed{29{,}607{,}600}$$

B) LET K BE THE NUMBER OF WAYS TO BREAK UP A INTO 5 NONEMPTY SUBSETS.

TO FORM A FUNCTION $f : A \to B$ THAT IS ONTO, WE CAN

1) BREAK UP A INTO 5 NONEMPTY SUBSETS, AND THEN

2) ASSIGN A DIFFERENT ELEMENT OF B TO EACH OF THE SUBSETS.

THEREFORE $K \cdot 5! = 5^{11} - \binom{5}{1}4^{11} + \binom{5}{2}3^{11} - \binom{5}{3}2^{11} + \binom{5}{4}1^{11}$,

$$\text{SO} \quad K = \boxed{\frac{5^{11} - \binom{5}{1}4^{11} + \binom{5}{2}3^{11} - \binom{5}{3}2^{11} + \binom{5}{4}}{5!}} = \boxed{246{,}730}$$

REMARK THIS NUMBER IS DENOTED BY $S(11,5)$ AND IS CALLED A STIRLING NUMBER OF THE SECOND KIND.

(6) NUMBER THE SOLDIERS $1, \ldots, 6$, AND

LET S BE THE SET OF PERMUTATIONS OF $\{1, \ldots, 6\}$.

LET A_i BE THE PERMUTATIONS CONTAINING THE STRING $i, i+1, i+2$ FOR $1 \le i \le 4$.

THEN $|\overline{A_1} \cap \cdots \cap \overline{A_4}| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + |A_1 \cap \cdots \cap A_4|$

$$= \boxed{6! - \binom{4}{1}4! + \left[3 \cdot 3! + 3 \cdot 2!\right] - \left[2 \cdot 2! + 2 \cdot 1\right] + 1} = \boxed{643}$$

(SINCE $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_4| = 3!$,

$|A_1 \cap A_3| = |A_2 \cap A_4| = |A_1 \cap A_4| = 2!$,

$|A_1 \cap A_2 \cap A_3| = |A_2 \cap A_3 \cap A_4| = 2!$, AND

$|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = 1$.)

MATH 145 CH. 7 – SOLUTIONS

(1) b)

n	0	1	2	3	4	5	6	7	8	9	
f_n	0	1	1	2	3	5	8	13	21	34	...

$f_0 + f_2 = 1$, $f_0 + f_2 + f_4 = 4$, $f_0 + f_2 + f_4 + f_6 = 12$

CONJECTURE: $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$

PF 1) THIS IS TRUE FOR $n = 0$, SINCE $f_0 = 0 = 1 - 1 = f_1 - 1$

2) LET $f_0 + f_2 + \cdots + f_{2n} = f_{2n+1} - 1$ FOR SOME INTEGER $n \geq 0$.

THEN $f_0 + f_2 + \cdots + f_{2n} + f_{2n+2} = (f_{2n+1} - 1) + f_{2n+2}$

$= (f_{2n+1} + f_{2n+2}) - 1 = f_{2n+3} - 1$.

THEREFORE $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$ FOR ALL INTEGERS $n \geq 0$ BY INDUCTION.

(8) a) 1) IF THE FIRST SQUARE IS BLUE, THEN THERE ARE h_{n-1} COLORINGS.

$\underset{1}{\underline{B}}\ \underset{2}{\underline{\ }}\ \underset{3}{\underline{\ }}\ \underset{4}{\underline{\ }}\ \cdots\ \underset{n}{\underline{\ }}$

2) IF THE FIRST SQUARE IS RED, THEN THERE ARE h_{n-2} COLORINGS,

$\underset{1}{\underline{R}}\ \underset{2}{\underline{B}}\ \underset{3}{\underline{\ }}\ \underset{4}{\underline{\ }}\ \cdots\ \underset{n}{\underline{\ }}$

THEREFORE $h_n = h_{n-1} + h_{n-2}$; WITH $h_1 = 2$ AND $h_2 = 3$.

SINCE (h_n) SATISFIES THE SAME RECURRENCE AS THE FIBONACCI NUMBERS AND $h_1 = f_3$ AND $h_2 = f_4$, WE HAVE $\boxed{h_n = f_{n+2}}$ FOR ALL n.

(34) $h_n = 8h_{n-1} - 16h_{n-2}$ FOR $n \geq 2$, $h_0 = -1$ AND $h_1 = 0$.

$r^2 = 8r - 16$ SO $r^2 - 8r + 16 = 0$ $(r-4)^2 = 0$ SO $r = 4$

$h_n = d(4^n) + e(n4^n)$ IS THE GENERAL SOL.

SINCE $h_0 = -1$, $d = -1$ AND SINCE $h_1 = 0$, $4d + 4e = 0$ SO $4e = 4$ AND $e = 1$

$\boxed{h_n = -4^n + n4^n = (n-1)4^n}$

(40) a) 1) IF THE FIRST DIGIT IS 2, THERE ARE a_{n-1} WAYS TO FINISH THE STRING.

$\underset{1}{\underline{2}}\ \underset{2}{\underline{\ }}\ \underset{3}{\underline{\ }}\ \cdots\ \underset{n}{\underline{\ }}$

2) IF THE FIRST DIGIT IS 0 OR 1, THEN THE SECOND DIGIT IS 2 AND THERE ARE a_{n-2} WAYS TO FINISH THE STRING.

$\underset{1}{\underline{0,1}}\ \underset{2}{\underline{2}}\ \underset{3}{\underline{\ }}\ \underset{4}{\underline{\ }}\ \cdots\ \underset{n}{\underline{\ }}$

THEREFORE $a_n = a_{n-1} + 2a_{n-2}$ FOR $n \geq 2$, WITH $a_0 = 1$ AND $a_1 = 3$.

b) $r^2 = r + 2$ GIVES $r^2 - r - 2 = 0$, $(r-2)(r+1) = 0$, $r = 2$ OR $r = -1$.

THEN $a_n = d(2^n) + e(-1)^n$ IS THE GENERAL SOL.

SINCE $a_0 = 1$, $d + e = 1$ AND SINCE $a_1 = 3$, $2d - e = 3$.

ADDING GIVES $3d = 4$, SO $d = 4/3$ AND $e = -1/3$.

THEN $\boxed{a_n = \frac{4}{3}(2^n) - \frac{1}{3}(-1)^n = \frac{1}{3}\left(2^{n+2} + (-1)^{n+1}\right)}$