MATH 145 SOLUTIONS - YELLOW SHEET

(1) $g(x) = (1 + x + \cdots + x^6)(1 + x + \cdots + x^{10})(1 + x^2 + x^4 + \cdots + x^{10})$

$$= \frac{1-x^7}{1-x} \cdot \frac{1-x^{11}}{1-x} \cdot \frac{1-x^{12}}{1-x^2}$$ ← SINCE WE WANT THE COEFF. OF $x^{10}$, WE CAN DISCARD TERMS WITH LARGER EXPONENTS:

$$\frac{1-x^7}{1-x} \cdot \frac{1}{1-x} \cdot \frac{1}{1-x^2} = \frac{(1-x^7)\cdot(1+x)(1+x)}{(1-x^2)(1-x^2)(1-x^2)}$$ ← (MULTIPLYING BY $\frac{(1+x)(1+x)}{(1+x)(1+x)}$)

$$= \frac{(1-x^7)(1+2x+x^2)}{(1-x^2)^3} = (1-x^7)(1+2x+x^2)(1-x^2)^{-3}$$

$$= (1 + \cancel{2x} + x^2 - \cancel{x^7} - 2x^8 - \cancel{x^9}) \sum_{n=0}^{\infty} \binom{n+2}{2} x^{2n}$$ ← (REPLACING $x$ BY $x^2$ IN THE SERIES FOR $(1-x)^{-3}$)

COEFF. OF $x^{10}$: $\boxed{\binom{7}{2} + \binom{6}{2} - 2\binom{3}{2}}$

(WE CAN DISCARD THE ODD POWERS IN THE FIRST FACTOR, SINCE THE SERIES ONLY HAS EVEN POWERS.)

(2) $g(x) = (x + x^2 + \cdots + x^5)(x^2 + x^3 + x^4 + \cdots)^5$ ← (SINCE WE WANT THE COEFF. OF $x^{20}$, WE CAN INCLUDE TERMS WITH EXPONENTS GREATER THAN 20)

$$= x(1 + x + \cdots + x^4)\left(x^2(1 + x + x^2 + \cdots)\right)^5$$

$$= x \cdot \frac{1-x^5}{1-x} \cdot x^{10} \cdot \frac{1}{(1-x)^5} = x^{11}(1-x^5)(1-x)^{-6}$$

$$= (x^{11} - x^{16}) \sum_{n=0}^{\infty} \binom{n+5}{5} x^n$$

COEFF. OF $x^{20}$: $\boxed{\binom{14}{5} - \binom{9}{5}}$

(3) $g(x) = (1 + x^2 + x^4 + \cdots + x^{50})(x + x^3 + x^5 + \cdots + x^{43})(1 + x + \cdots + x^{29})(1 + x + \cdots + x^{24})$

$$= \frac{1-x^{52}}{1-x^2} \cdot x(1 + x^2 + x^4 + \cdots + x^{42}) \cdot \frac{1-x^{30}}{1-x} \cdot \frac{1-x^{25}}{1-x}$$

$$= \frac{1-x^{52}}{1-x^2} \cdot x \cdot \frac{1-x^{44}}{1-x^2} \cdot \frac{1-x^{30}}{1-x} \cdot \frac{1-x^{25}}{1-x}$$ ← (SINCE WE WANT THE COEFF. OF $x^{50}$, WE CAN DISCARD TERMS WITH LARGER EXPONENTS)

$$\frac{1}{1-x^2} \cdot x \cdot \frac{1-x^{44}}{1-x^2} \cdot \frac{1-x^{30}}{1-x} \cdot \frac{1-x^{25}}{1-x}$$ ← (MULTIPLY BY $\frac{(1+x)(1+x)}{(1+x)(1+x)}$)

$$= \frac{x(1-x^{44})(1-x^{30})(1-x^{25})(1+2x+x^2)}{(1-x^2)^4}$$

$$(x + 2x^2 + x^3)(1 - x^{44} - x^{30} - x^{25})(1-x^2)^{-4}$$

$$= (x + 2x^2 + x^3)(1 - x^{44} - x^{30} - x^{25}) \sum_{n=0}^{\infty} \binom{n+3}{3} x^{2n}$$ ← (REPLACING $x$ BY $x^2$ IN THE SERIES FOR $(1-x)^{-4}$)

COEFF. OF $x^{50}$ IN $(2x^2 - 2x^{46} - 2x^{32} - x^{26} - x^{28}) \sum_{n=0}^{\infty} \binom{n+3}{3} x^{2n}$ ← (WE CAN DISCARD THE ODD POWERS IN THE FIRST FACTOR, SINCE THE SERIES HAS ONLY EVEN POWERS)

IS GIVEN BY $\boxed{2\binom{27}{3} - 2\binom{5}{3} - 2\binom{12}{3} - \binom{15}{3} - \binom{14}{3}}$

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(4) $2x + y + z = n$ WITH $x, y \geq 0$ AND $z \geq 1$

$$g(x) = (1 + x^2 + x^4 + x^6 + \dots)(1 + x + x^2 + x^3 + \dots)(x + x^2 + x^3 + \dots)$$

$$= \frac{1}{1-x^2} \cdot \frac{1}{1-x} \cdot \frac{x}{1-x} = \frac{x}{(1+x)(1-x)^3}$$

$$\frac{x}{(1+x)(1-x)^3} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} + \frac{D}{(1-x)^3}$$

$$x = A(1-x)^3 + B(1+x)(1-x)^2 + C(1+x)(1-x) + D(1+x)$$

$x=1$: $1 = 2D$ $D = 1/2$

$x=-1$: $-1 = 8A$ $A = -1/8$

$x^3$ coeff.: $0 = -A + B$ $B = -1/8$

$x=0$: $0 = A + B + C + D = C + 1/4$ so $C = -1/4$

THEN $g(x) = -\frac{1}{8} \cdot \frac{1}{1+x} - \frac{1}{8} \cdot \frac{1}{1-x} - \frac{1}{4} \cdot \frac{1}{(1-x)^2} + \frac{1}{2} \cdot \frac{1}{(1-x)^3}$

$$= -\frac{1}{8}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{1}{8}\sum_{n=0}^{\infty} x^n - \frac{1}{4}\sum_{n=0}^{\infty}\binom{n+1}{1}x^n + \frac{1}{2}\sum_{n=0}^{\infty}\binom{n+2}{2}x^n$$

$$= \sum_{n=0}^{\infty}\left[-\frac{1}{8}(-1)^n - \frac{1}{8} - \frac{1}{4}(n+1) + \frac{1}{2}\binom{n+2}{2}\right]x^n,$$

so $a_n = \boxed{\frac{1}{2}\binom{n+2}{2} - \frac{1}{4}(n+1) - \frac{1}{8} - \frac{1}{8}(-1)^n}$

$= \boxed{\frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{8} + \frac{1}{8}(-1)^{n+1}}$

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OR use $g(x) = \frac{x}{(1-x^2)(1-x)^2} \cdot \frac{(1+x)^2}{(1+x)^2} = x(1+2x+x^2)(1-x^2)^{-3}$

$$= (x + 2x^2 + x^3)\sum_{m=0}^{\infty}\binom{m+2}{2}x^{2m},$$

AND THEN FIND THE COEFF. OF $x^n$ TO GET

1) $a_n = 2\binom{m+2}{2} = \boxed{\frac{n(n+2)}{4}}$ IF $n$ IS EVEN

2) $a_n = \binom{\frac{n+3}{2}}{2} + \binom{\frac{n+1}{2}}{2} = \boxed{\frac{(n+1)^2}{4}}$ IF $n$ IS ODD

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(5)

$$g_e(x) = \underbrace{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^3}_{(1,2,3)} \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}_{(4)}$$

$$= \left(\frac{e^x - e^{-x}}{2}\right)^3 (e^x)$$

$$= \frac{1}{8}\left(e^{3x} - 3e^x + 3e^{-x} - e^{-3x}\right)(e^x)$$

$$= \frac{1}{8}\left(e^{4x} - 3e^{2x} + 3 - e^{-2x}\right)$$

$\longleftarrow$ (NOTICE THAT $x=0$ GIVES 0) SO $a_0 = 0$

$$= \sum_{n=1}^{\infty} \frac{1}{8}\left[4^n - 3\cdot 2^n - (-2)^n\right] \cdot \frac{x^n}{n!},$$

so $a_n = \boxed{\frac{1}{8}\left(4^n - 3\cdot 2^n - (-2)^n\right)}$ FOR $n \geq 1$

(6)

$$g_e(x) = \underbrace{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2}_{(1,2)} \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2}_{(3,4)} \underbrace{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}_{(5)}$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 (e^x)^2 (e^x - 1)$$

$$= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)(e^{2x})(e^x - 1)$$

$$= \frac{1}{4}\left(e^{4x} + 2e^{2x} + 1\right)(e^x - 1)$$

$\longleftarrow$ (NOTICE THAT $x=0$ GIVES 0, SO $a_0 = 0$)

$$= \frac{1}{4}\left(e^{5x} + 2e^{3x} + e^x - e^{4x} - 2e^{2x} - 1\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{4}\left(5^n + 2\cdot 3^n + 1 - 4^n - 2\cdot 2^n\right) \cdot \frac{x^n}{n!},$$

so $a_n = \boxed{\frac{1}{4}\left(5^n + 2\cdot 3^n + 1 - 4^n - 2^{n+1}\right)}$ FOR $n \geq 1$

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(26)
$$g_e(x) = \underbrace{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2}_{(A,\,G)} \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2}_{(B,\,O)}$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 (e^x)^2 = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)\left(e^{2x}\right)$$

$$= \underline{\frac{1}{4}\left(e^{4x} + 2e^{2x} + 1\right)} = \frac{1}{4}\left[\sum_{n=0}^{\infty} \frac{4^n x^n}{n!} + 2\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} + 1\right]$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{4}\left(4^n + 2 \cdot 2^n\right) \frac{x^n}{n!},$$ so $\underline{a_n} = \frac{1}{4}\left(4^n + 2^{n+1}\right) = \boxed{4^{n-1} + 2^{n-1}}$, $n \ge 1$

(27)
$$g_e(x) = \underbrace{\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)^2}_{(1,\,3)} \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3}_{(5,\,7,\,9)}$$

$$= \left(\frac{e^x + e^{-x}}{2} - 1\right)^2 (e^x)^3 = \frac{1}{4}\left(e^x + e^{-x} - 2\right)^2 \cdot e^{3x}$$

$$= \frac{1}{4}\left(e^{2x} + e^{-2x} + \underline{4} - 4e^x - 4e^{-x} + \underline{2}\right)e^{3x}$$

$$= \underline{\frac{1}{4}\left(e^{5x} + e^x + 6e^{3x} - 4e^{4x} - 4e^{2x}\right)},$$

so $a_n = \boxed{\frac{1}{4}\left(5^n + 1 + 6(3^n) - 4(4^n) - 4(2^n)\right)}$

$= \boxed{\frac{1}{4}\left(5^n + 1 + 6(3^n) - 4^{n+1} - 2^{n+2}\right)}$

(28)
$$g_e(x) = \underbrace{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2}_{(4,\,6)} \underbrace{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2}_{(5,\,7)} \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2}_{(8,\,9)}$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 (e^x - 1)^2 (e^x)^2$$

$$= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)\left(e^{2x} - 2e^x + 1\right)\left(e^{2x}\right)$$

$$= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)\left(e^{4x} - 2e^{3x} + e^{2x}\right)$$

$$= \frac{1}{4}\left(e^{6x} - 2e^{5x} + 3e^{4x} - 4e^{3x} + 3e^{2x} - 2e^x + 1\right)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{4}\left(6^n - 2 \cdot 5^n + 3 \cdot 4^n - 4 \cdot 3^n + 3 \cdot 2^n - 2\right) \cdot \frac{x^n}{n!},$$

so $a_n = \boxed{\frac{1}{4}\left(6^n - 2(5^n) + 3(4^n) - 4(3^n) + 3(2^n) - 2\right)}$ FOR $n \ge 1$