MATH 145

CH. 3 – SOLUTIONS

(7) DENOTE THE INTEGERS BY $a_1, \dots, a_{52}$, AND MAP EACH INTEGER TO THE SET AMONG $S_0 = \{0\}$, $S_1 = \{1, 99\}$, $S_2 = \{2, 98\}$, $\dots$, $S_{49} = \{49, 51\}$, $S_{50} = \{50\}$ WHICH CONTAINS ITS REMAINDER WHEN DIVIDED BY 100. SINCE THERE ARE 52 INTEGERS AND ONLY 51 SETS, TWO INTEGERS a_i AND a_j MUST BE MAPPED TO THE SAME SET BY THE PH PRINCIPLE.

1) IF a_i AND a_j HAVE REMAINDER r WHEN DIVIDED BY 100, THEN $a_i - a_j$ IS DIVISIBLE BY 100.

2) IF a_i HAS REMAINDER r AND a_j HAS REMAINDER $100 - r$ WHEN DIVIDED BY 100, THEN $a_i + a_j$ IS DIVISIBLE BY 100.

(9) (FOR 9 PEOPLE INSTEAD OF 10)

LET S BE THE SET OF ALL SUBSETS OF THE PEOPLE, OTHER THAN THE EMPTY SET AND THE SET OF ALL 9 PEOPLE. THEN S HAS $2^9 - 2 = 510$ ELEMENTS, AND WE CAN MAP EACH SUBSET IN S TO ITS AGE SUM IN $\{1, \dots, 480\}$. BY THE PH PRINCIPLE, TWO ELEMENTS OF S HAVE THE SAME AGE SUM; AND WE CAN LEAVE OUT ANY PEOPLE IN BOTH SUBSETS.

OR a) IF 2 PEOPLE P_1 AND P_2 HAVE THE SAME AGE, THEN WE CAN TAKE THE 2 GROUPS TO BE $\{P_1\}$ AND $\{P_2\}$.

b) IF NO 2 PEOPLE HAVE THE SAME AGE, THEN THERE ARE $2^9 - 1 = 511$ NONEMPTY SUBSETS OF THE PEOPLE; AND WE CAN MAP EACH SUBSET TO ITS AGE SUM IN $\{1, \dots, 504\}$ (SINCE $52 + 53 + \dots + 60 = 504$). BY THE PH PRINCIPLE, TWO ELEMENTS OF S HAVE THE SAME AGE SUM; AND WE CAN OMIT ANY PEOPLE COMMON TO BOTH SUBSETS.

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(17) a) IF THERE ARE 3 OR MORE PEOPLE WITH 0 ACQUAINTANCES, THEN WE ARE DONE.

b) IF 0 OR 1 PERSON HAS 0 ACQUAINTANCES, THEN THE OTHER 99 OR 100 PEOPLE HAVE POSSIBLE NUMBER OF ACQUAINTANCES IN $\{2, 4, 6, \dots, 98\}$. SINCE THERE ARE 49 BOXES AND MORE THAN 98 PEOPLE, AT LEAST 3 PEOPLE ARE MAPPED TO THE SAME BOX.

c) IF THERE ARE 2 PEOPLE WITH 0 ACQUAINTANCES, THEN THE OTHER 98 PEOPLE HAVE POSSIBLE NUMBER OF ACQUAINTANCES IN $\{2, 4, 6, \dots, 96\}$. SINCE THERE ARE 48 BOXES AND MORE THAN 96 PEOPLE, AT LEAST 3 PEOPLE ARE MAPPED TO THE SAME BOX.

OR MAP EACH OF THE 100 PEOPLE TO THEIR NUMBER OF ACQUAINTANCES IN $\{0, 2, 4, \dots, 98\}$. SINCE THERE ARE 100 PEOPLE AND 50 BOXES, SOME BOX WILL CORRESPOND TO 3 OR MORE PEOPLE UNLESS 2 PEOPLE GET MAPPED TO EVERY BOX (SINCE IF A BOX CORRESPONDS TO 0 OR 1 PEOPLE, THE OTHER 100 OR 99 PEOPLE ARE GETTING MAPPED TO 49 BOXES). HOWEVER, IF 2 PEOPLE HAVE 0 ACQUAINTANCES, THEN NOBODY CAN HAVE MORE THAN 96 ACQUAINTANCES; SO IT IS IMPOSSIBLE THAT 2 PEOPLE ARE MAPPED TO EVERY BOX.

MATH 145 — CH. 11 – SOLUTIONS

(3) THERE IS NO GRAPH WITH DEGREE SEQUENCE $(4,4,3,2,2)$, SINCE THE SUM OF THE DEGREES MUST BE EVEN (OR EQUIVALENTLY, SINCE THE NUMBER OF VERTICES OF ODD DEGREE MUST BE EVEN).

(5) i) LET $\{v_1, \ldots, v_n\}$ BE THE VERTICES, AND MAP EACH VERTEX TO ITS DEGREE.

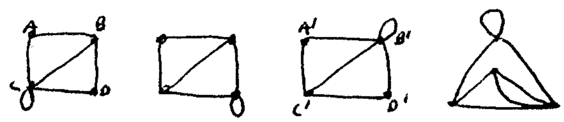
a) IF NO VERTEX HAS DEGREE 0, THEN THE POSSIBLE DEGREES ARE $\{1, \ldots, n-1\}$; SO BY THE PH PRINCIPLE, THERE ARE 2 VERTICES OF THE SAME DEGREE.

b) IF SOME VERTEX HAS DEGREE 0, THEN THE POSSIBLE DEGREES ARE $\{0, \ldots, n-2\}$; SO BY THE PH PRINCIPLE, THERE ARE 2 VERTICES OF THE SAME DEGREE.

ii) THIS DOES NOT HOLD FOR MULTIGRAPHS; FOR EXAMPLE, THE FOLLOWING MULTIGRAPH DOES NOT HAVE 2 VERTICES OF THE SAME DEGREE:

REMARK THIS IS EQUIVALENT TO #16 IN CH. 3.

(11)



a) THE 4TH GRAPH IS NOT ISOMORPHIC TO THE OTHERS, SINCE IT HAS 2 EDGES BETWEEN A PAIR OF VERTICES.

b) THE 2ND GRAPH IS NOT ISOMORPHIC TO THE 1ST OR 3RD, SINCE IT HAS A VERTEX OF DEGREE 4 AND THE 1ST AND 3RD GRAPHS DO NOT.

c) MAPPING $C \to B'$, $B \to C'$, $A \to D'$, $D \to A'$ GIVES AN ISOMORPHISM OF GRAPHS 1 AND 3.

(13) LET $v_0 v_1 \ldots v_n$ BE A WALK OF SHORTEST LENGTH JOINING v_0 AND v_n.

IF $v_i = v_j$ FOR SOME $i < j$, THEN

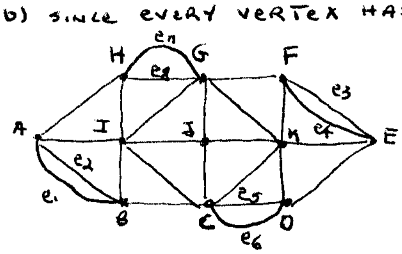
a) $v_0 \ldots v_i v_{j+1} \ldots v_n$ WOULD BE A SHORTER WALK IF $j < n$, AND

b) $v_0 \ldots v_i$ WOULD BE A SHORTER WALK IF $j = n$.

THEREFORE $v_i \neq v_j$ IF $i < j$, SO $v_0 v_1 \ldots v_n$ IS A PATH.

(29) a) THE FIRST GRAPH DOES NOT HAVE AN EULERIAN TRAIL, SINCE IT HAS MORE THAN 2 VERTICES WITH ODD DEGREE.

b) SINCE EVERY VERTEX HAS EVEN DEGREE IN THE 2ND GRAPH, IT HAS A CLOSED EULERIAN TRAIL.



1) FIRST WE CONSTRUCT A CLOSED TRAIL:

$$A \xrightarrow{e_2} B \to C \xrightarrow{e_5} D \to E \xrightarrow{e_3} F \to G \xrightarrow{e_8} H \to A$$

2) DELETE THE EDGES USED SO FAR, AND START WITH A NEW EDGE WHICH HAS A VERTEX IN THE FIRST TRAIL AND CONSTRUCT A NEW CLOSED TRAIL:

$D \xrightarrow{e_6} C \to K$

$$A \xrightarrow{e_1} B \to I \to H \xrightarrow{e_7} G \to K \to F \xrightarrow{e_4} E \to K \to J \to I \to A$$

3) GLUE THESE TRAILS TOGETHER TO GET A LONGER CLOSED TRAIL:

$$A \xrightarrow{e_2} B \to C \xrightarrow{e_5} D \to E \xrightarrow{e_3} F \to G \xrightarrow{e_8} H \to A \xrightarrow{e_1} B \to I \to H \xrightarrow{e_7} G \to K \to F \xrightarrow{e_4} E \to K \to J \to I \to A$$

$D \xrightarrow{e_6} C \to K$

4) REPEAT STEP 2 TO GET A NEW CLOSED TRAIL:

$$I \to G \to J \to C \to I$$

5) COMBINE THIS TRAIL WITH THE PREVIOUS ONE:

$$A \xrightarrow{e_2} B \to C \xrightarrow{e_5} D \to E \xrightarrow{e_3} F \to G \xrightarrow{e_8} H \to A \xrightarrow{e_1} B \to I \to G \to J \to C \to I$$
$$\to H \xrightarrow{e_7} G \to K \to F \xrightarrow{e_4} E \to K \to D \xrightarrow{e_6} C \to K \to J \to I \to A$$

(30) EVERY VERTEX OF K_n HAS DEGREE $n-1$, SO

a) K_n HAS A CLOSED EULERIAN TRAIL IFF n IS ODD (SO EVERY VERTEX HAS EVEN DEGREE)

b) K_n HAS AN OPEN EULERIAN TRAIL IFF $n = 2$ (SO THERE ARE EXACTLY 2 VERTICES OF ODD DEGREE)