MATH 145 — ANSWERS – TEST 1 REVIEW PROBLEMS

① $\boxed{\underbrace{\binom{9}{4}}_{E's}\underbrace{\binom{5}{2}}_{N's}\underbrace{\binom{3}{2}}_{S's}} = \boxed{\dfrac{9!}{4!\,2!\,2!\,1!}}$ (NUMBER OF PERMUTATIONS OF $S = \{4\cdot E, 2\cdot N, 2\cdot S, 1\cdot T\}$)

② THE UNITS DIGIT MUST BE ODD, AND THE LEADING DIGIT CANNOT BE THE SAME AS THE UNITS DIGIT AND CANNOT BE 0!

a) 1-DIGIT: $\underline{5}$ b) 2-DIGIT: $\underline{8}\cdot\underline{5}$ c) 3-DIGIT: $\underline{8}\cdot\underline{8}\cdot\underline{5}$ d) 4-DIGIT: $\underline{8}\cdot\underline{8}\cdot\underline{7}\cdot\underline{5}$

TOTAL: $\boxed{5 + 8\cdot 5 + 8\cdot 8\cdot 5 + 8\cdot 8\cdot 7\cdot 5} = \boxed{2605}$

③ a) $6\cdot 6\cdot 6\cdot 6 = \boxed{6^4}$ b) $\boxed{6\cdot 5\cdot 4\cdot 3}$ c) $\boxed{\binom{6}{4}}$

④ a) 1) ARRANGE THE TURKS: 30! WAYS

2) CHOOSE A GAP FOR THE GREEKS: 31 WAYS

3) ARRANGE THE GREEKS: 10! WAYS

ANSWER: $\boxed{(30!)\cdot(31)\cdot(10!)}$

OR 1) ARRANGE 31 "PEOPLE" (COUNTING THE GREEKS AS 1 PERSON): 31! WAYS

2) ARRANGE THE GREEKS: 10! WAYS

ANSWER: $\boxed{(31!)(10!)}$

b) 1) ARRANGE THE TURKS IN A CIRCLE: 29! WAYS

2) SELECT THE GAPS FOR THE GREEKS: $\binom{30}{10}$ WAYS

3) ARRANGE THE GREEKS IN THE GAPS: 10! WAYS

ANSWER: $\boxed{(29!)\binom{30}{10}(10!)}$

⑤ A) 1) TOTAL NUMBER OF COMMITTEES: $\binom{40}{8}$

2) COMMITTEES WITH NO MEN: $\binom{25}{8}$

ANSWER: $\boxed{\binom{40}{8} - \binom{25}{8}}$

B) 1) COMMITTEES WITH AT LEAST 6 WOMEN: $\underbrace{\binom{15}{2}\binom{25}{6}}_{2M,\,6W} + \underbrace{\binom{15}{1}\binom{25}{7}}_{1M,\,7W} + \underbrace{\binom{25}{8}}_{8W}$

2) COMMITTEES WITH AT LEAST 6 WOMEN, AND INCLUDING JIM AND MOLLY: $\underbrace{\binom{14}{1}\binom{24}{5}}_{+1M,\,+5W} + \underbrace{\binom{24}{6}}_{+6W}$

ANSWER: $\boxed{\binom{15}{2}\binom{25}{6} + \binom{15}{1}\binom{25}{7} + \binom{25}{8} - \binom{14}{1}\binom{24}{5} - \binom{24}{6}}$

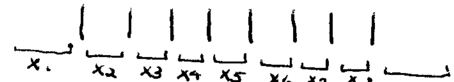
⑥ A) IF WE LINE UP THE 24 STICKS NOT CHOSEN, THERE ARE 25 GAPS; SO WE NEED TO SELECT 8 OF THESE IN WHICH TO PUT THE CHOSEN STICKS.

ANSWER: $\boxed{\binom{25}{8}}$

OR LINE UP THE 8 CHOSEN STICKS, AND REMOVE 7 BLOCKERS TO BE INSERTED AT THE END. THIS LEAVES 17 REMAINING STICKS, AND THESE STICKS AND THE 8 CHOSEN STICKS CAN BE ARRANGED IN $\boxed{\binom{25}{8}}$ WAYS

B) LINE UP THE 8 CHOSEN STICKS, AND REMOVE $2\cdot 7 = 14$ BLOCKERS TO BE INSERTED AT THE END. THIS LEAVES 10 ADDITIONAL STICKS, AND THESE STICKS AND THE 8 CHOSEN STICKS CAN BE ARRANGED IN $\boxed{\binom{18}{8}}$ WAYS

OR FIND THE NUMBER OF SOLUTIONS OF $x_1 + \cdots + x_9 = 24$ WITH $x_1, x_9 \geq 0$ AND $x_2, \ldots, x_8 \geq 2$.



MATH 145 ANSWERS – TEST 1 REVIEW PROBLEMS

(7) LET $S = \{1, \ldots, 60000\}$, $A_1 = \{n \in S : 4 \mid n\}$, $A_2 = \{n \in S : 6 \mid n\}$, $A_3 = \{n \in S : 10 \mid n\}$,

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3|$$

$$= 60{,}000 - \left\lfloor \frac{60{,}000}{4} \right\rfloor - \left\lfloor \frac{60{,}000}{6} \right\rfloor - \left\lfloor \frac{60{,}000}{10} \right\rfloor + \left\lfloor \frac{60{,}000}{12} \right\rfloor + \left\lfloor \frac{60{,}000}{20} \right\rfloor + \left\lfloor \frac{60{,}000}{30} \right\rfloor - \left\lfloor \frac{60{,}000}{60} \right\rfloor$$

$$= \boxed{60{,}000 - 15{,}000 - 10{,}000 - 6{,}000 + 5{,}000 + 3{,}000 + 2{,}000 - 1{,}000} = \boxed{38{,}000}$$

(8) A) SINCE $\underline{(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k}$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^k = \sum_{k=0}^{n} \binom{n}{k} (-3)^k = (1+(-3))^n = \boxed{(-2)^n}$

B) SINCE $\underline{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$, $\underbrace{\binom{24}{15} + \binom{24}{14}} + \binom{25}{14} = \underbrace{\binom{25}{15}} + \binom{25}{14} = \boxed{\binom{26}{15}}$

(9) LET $\underline{Y_1 = X_1 - 2,\ Y_2 = X_2 + 1,\ Y_3 = X_3 - 3,\ Y_4 = X_4 - 2}$ TO GET

$\underline{Y_1 + Y_2 + Y_3 + Y_4 = 20}$ WITH $0 \le Y_1 \le 6$, $0 \le Y_2 \le 5$, $0 \le Y_3 \le 4$, $0 \le Y_4 \le 14$.

LET S BE THE SET OF ALL NONNEGATIVE INTEGER SOLUTIONS, AND LET

A_1 BE THE SOL. IN S WITH $\underline{Y_1 \ge 7}$, A_2 BE THE SOL. IN S WITH $\underline{Y_2 \ge 6}$,

A_3 BE THE SOL. IN S WITH $\underline{Y_3 \ge 5}$, A_4 BE THE SOL. IN S WITH $\underline{Y_4 \ge 15}$.

$$|\overline{A_1} \cap \cdots \cap \overline{A_4}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + |A_1 \cap \cdots \cap A_4|$$

$$= \boxed{\underbrace{\binom{23}{3}}_{|S|} - \underbrace{\binom{16}{3}}_{|A_1|} - \underbrace{\binom{17}{3}}_{|A_2|} - \underbrace{\binom{18}{3}}_{|A_3|} - \underbrace{\binom{8}{3}}_{|A_4|} + \underbrace{\binom{10}{3}}_{1,2} + \underbrace{\binom{11}{3}}_{1,3} + \underbrace{\binom{12}{3}}_{2,3} + \underbrace{\binom{3}{3}}_{3,4} - \underbrace{\binom{5}{3}}_{1,2,3}}$$

(10) LET S BE THE SET OF ALL POSSIBLE TEAM ASSIGNMENTS, AND

LET A_i BE THE SET OF ASSIGNMENTS FOR WHICH TEAM i IS EMPTY, FOR $1 \le i \le 4$.

$$|\overline{A_1} \cap \cdots \cap \overline{A_4}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + |A_1 \cap \cdots \cap A_4|$$

$$= \boxed{4^{15} - \binom{4}{1} 3^{15} + \binom{4}{2} 2^{15} - \binom{4}{3}}$$

(11) A) $\underline{\text{IF THE FIRST LETTER IS } b \text{ OR } c}$, THERE ARE 2 CHOICES FOR THE FIRST LETTER AND THEN h_{n-1} WAYS TO FINISH THE WORD, GIVING $\underline{2h_{n-1}}$ POSSIBILITIES.

B) $\underline{\text{IF THE FIRST LETTER IS } a}$, THERE ARE h_{n-1} WAYS TO CHOOSE THE REMAINING LETTERS IF THERE WERE NO RESTRICTIONS, BUT WE MUST SUBTRACT THE CODEWORDS WHICH WOULD START WITH $\underline{abc}$ OR $\underline{acb}$.

$\underset{1}{\underline{a}}\ \underset{2}{\underline{\ \ }}\ \underset{3}{\underline{\ \ }}\ \underset{4}{\underline{\ \ }}\ \cdots\ \underset{n}{\underline{\ \ }}$

SINCE THERE ARE h_{n-3} CODEWORDS STARTING WITH $\underline{abc}$ AND SIMILARLY FOR $\underline{acb}$,

THIS GIVES $\underline{h_{n-1} - 2h_{n-3}}$ POSSIBILITIES.

THEREFORE $\boxed{h_n = 3h_{n-1} - 2h_{n-3}}$

(12) 1) THERE ARE $\binom{7}{3}$ WAYS TO CHOOSE THE 3 INTEGERS TO BE LEFT FIXED.

2) THERE ARE D_4 WAYS TO ARRANGE THE REMAINING 4 INTEGERS SO THAT NONE OF THEM IS LEFT IN ITS NATURAL POSITION.

$\underline{\text{ANSWER}}$: $\binom{7}{3} D_4 = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} \cdot 9 = 35 \cdot 9 = \boxed{315}$ SINCE $D_4 = 4D_3 + (-1)^4 = 4 \cdot 2 + 1 = 9$.