1. Let $G$ be a graph with $n$ vertices.
   a) If $G$ is connected, what is its minimum number of edges?
   b) If $G$ is not connected, what is its maximum number of edges?

2. Find the following, and justify your answers:
   a) A graph that has a Hamilton cycle but does not have a closed Eulerian trail.
   b) A graph that has a closed Eulerian trail but does not have a Hamilton cycle.

3. a) Show that there is no graph with degree sequence $(6, 5, 4, 4, 3, 1, 1)$.
    b) Draw a general graph with degree sequence $(6, 5, 4, 4, 3, 1, 1)$.

4. Prove that a graph $G = (V, E)$ is not connected iff there is a subset $U$ of $V$
   (with $U \neq \emptyset$, $U \neq V$) such that $xy \in E$ whenever $x \notin U$ and $y \notin U$.

5. For $n \geq 3$, let $G_n$ be the graph obtained from $K_n$ by deleting an edge.
   Find the values of $n$ for which $G_n$ has an Eulerian trail, and justify your answer.

6. In the following, use the definition that a tree is a connected graph
   which has no cycles:
   a) Prove that a graph $G$ is a tree iff $G$ is connected, and every edge of $G$
      is a bridge.
   b) Prove that a graph $G$ is a tree iff every pair of distinct vertices is
      connected by a unique path.

7. Draw a tree with degree sequence $(5, 3, 3, 3, 1, 1, 1, 1)$, or
   explain why this is impossible.

8. Suppose there are 12 people in a room. Show that either there is at least
   one person who knows at least 6 others, or there is a group of 3 people,
   none of whom know each other.

9. Suppose there are 10 users of Facebook.
   Assume that there are 10 users of Facebook.
   If user $k$ has $k$ friends for $1 \leq k \leq 9$, who are the friends of user 10?

10. Let $G$ be a graph with $2m$ vertices that does not contain a triangle $C_3$ as a
    subgraph. Use induction on $m$ to prove that $G$ has at most $m^2$ edges.

11. If $C_n$ is the cycle graph of order $n$,
    show that its chromatic polynomial is $(k-1)^n + (-1)^n (k-1)$. 