MATH 145 ANSWERS – TEST 2 REVIEW PROBLEMS

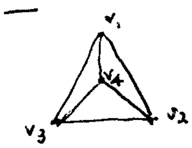
① A) A connected graph with n vertices has at least $\underline{n-1}$ edges (by Th. 11.5.1).

B) A graph with n vertices which is not connected has a maximum of $\underline{\binom{n-1}{2}}$ edges:

1) A graph consisting of K_{n-1} and an isolated vertex has n vertices and $\binom{n-1}{2}$ edges, and

2) A graph with n vertices and $\binom{n-1}{2}+1$ edges must be connected: if it had a component with m vertices where $m<n$, then it has at most $\binom{m}{2}+\binom{n-m}{2}$ edges, and $\binom{m}{2}+\binom{n-m}{2}\le\binom{n-1}{2}$.

② A) $\underline{K_4}$ has a Hamilton cycle but no closed Eulerian trail:

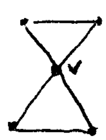


it has no closed Eulerian trail since it has vertices of odd degree, and $v_1 v_2 v_4 v_3 v_1$ is a Hamilton cycle.

B) $\underline{K_{2,4}}$ has a closed Eulerian trail but no Hamilton cycle:

it has a closed Eulerian trail since every vertex has even degree, but it has no Hamilton cycle since $|A| = 2 \neq 4 = |B|$.

$\boxed{\text{OR}}$



The "hourglass" graph G has a closed Eulerian trail since every vertex has even degree, but it does not have a Hamilton cycle since any cycle starting at v has length 3.

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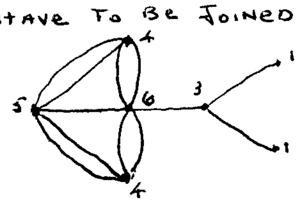
③ A) Let $(6,5,4,4,3,1,1)$ be the degree sequence of a graph. If we delete the vertex of degree 6, we get a graph with degree sequence $(4,3,3,2,0,0)$; and this is impossible since the vertex of degree 4 would have to be joined to one of the vertices of degree 0.

B)



④ G is not connected iff there is a subset U of V with $U \neq \emptyset$, $U \neq V$, and $xy \notin E$ whenever $x \in U$ and $y \notin U$.

$\underline{\text{PF}}$ $\Rightarrow$ Suppose G is not connected, and let G_1 be a connected component of G. If $U = V(G_1)$, then $U \neq \emptyset$ and $U \neq V$; and if $x \in U$ and $y \notin U$, then $xy \notin E$ since otherwise $y \in U$ (because G_1 is a maximal connected subgraph of G).

$\Leftarrow$ Suppose a subset U exists with this property, and let $x \in U$ and $y \notin U$. If there were a path $v_0 v_1 v_2 \ldots v_n$ from x to y, let v_i be the last vertex in U. Then v_i, v_{i+1} are adjacent with $v_i \in U$ and $v_{i+1} \notin U$; so this gives a contradiction. Therefore there is no path from x to y, so G is not connected.

Answers – Test 2 Review Problems

(5) G_n has 2 vertices of degree $n-2$ and $n-2$ vertices of degree $n-1$, so G_n has an open Eulerian trail if n is odd or $n=4$, since in both cases G_n has exactly 2 vertices of odd degree.

(6) A) G is a tree iff G is connected and every edge of G is a bridge.

PF $\Rightarrow$ Suppose G is a tree, so G is connected. If uv is an edge of G which is not a bridge, then $G-uv$ is connected; so there is a path from u to v in $G-uv$. Inserting uv into this path gives a cycle, and this contradicts that G has no cycles. Therefore every edge of G is a bridge.

$\Leftarrow$ Suppose G is connected, and every edge of G is a bridge. If G contains a cycle $uv_1v_2\ldots v_nvu$, then the graph $G-uv$ is not connected; so there are vertices $x, y \in G-uv$ which are not joined by a walk in $G-uv$. Since there is a walk from x to y in G, this walk must include uv; so replacing uv by $uv_1\ldots v_nv$ gives a walk from x to y in $G-uv$. This gives a contradiction, so G contains no cycles and therefore is a tree.

B) G is a tree iff every pair of vertices is connected by a unique path.

PF $\Leftarrow$ Suppose every pair of vertices is connected by a unique path; then G is connected. If G contains a cycle $uv_1\ldots v_nvu$, then there are 2 paths uv and $uv_1\ldots v_nv$ connecting u and v; so G has no cycles and hence is a tree.

$\Rightarrow$ Let G be a tree, and let u and v be vertices of G. Since G is connected, there is a path connecting u and v. If there are 2 paths P_1 and P_2 from u to v, let x be the last vertex before P_1 and P_2 break apart and let y be the first vertex where they join back together. Taking P_1 from x to y and then taking the reverse of P_2 from y to x gives a cycle, which contradicts the fact that G has no cycles. Therefore there is a unique path between every pair of distinct vertices.

(7) No tree can have degree sequence $(5,3,3,3,2,1,1,1,1)$; since there is a vertex of degree 5, there would have to be 5 leaves (by #62 in Ch. 11).

(8) Form a graph G by drawing an edge between 2 people iff they know each other. If there is no group of 3 people with the property that none of them know each other, then $\alpha(G) \le 2$ so $\chi(G) \ge \left\lceil \frac{13}{\alpha(G)} \right\rceil \ge \left\lceil \frac{13}{2} \right\rceil = 7$. Since $\chi(G) \le \Delta + 1$, $\Delta \ge 6$ so one person knows at least 6 others.

$\boxed{\text{OR}}$ Suppose no person knows at least 6 others. Let P_1 be any person, and let G_1 consist of P_1 and the people P_1 knows, so $|G_1| \le 6$. Let P_2 be a person not in G_1, and let G_2 consist of P_2 and the people P_2 knows, so $|G_2| \le 6$. Since there are 13 people, there is a person P_3 who is not in G_1 or G_2; so $\{P_1, P_2, P_3\}$ is a group of 3 people who do not know each other.

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(9) Since 9 knows everyone, 1 knows 9 and no one else.

Since 8 knows everyone except 1, 2 knows 9 and 8 and no one else.

Similarly, 7 knows everyone except 1 and 2,

6 knows everyone except 1, 2, and 3, and 5 knows everyone except 1 – 4.

Therefore users 5–9 are the friends of user 10.

(10) 1) If $m=1$, G has 2 vertices so it has at most 1 edge; so the assertion holds for $m=1$.

2) Assume that the statement is true for m, and let G be a graph with $2(m+1)$ vertices that doesn't contain a triangle as a subgraph.

Let $e = uv$ be an edge of G, and let G' be the graph obtained from G by deleting u and v. Then G' has $2m$ vertices and doesn't contain a triangle, so G' has at most m^2 edges (by the induction hypothesis).

If z is any vertex of G', it cannot be adjacent to both u and v (since G does not contain a triangle); so there is at most one edge in G that is not in G' for each vertex z in G', in addition to e itself.

Therefore G has at most $m^2 + (2m+1) = (m+1)^2$ edges, so the statement is true for $m+1$.

(11) (see #14, Ch. 12)

By induction on n:

1) This is true for $n=3$, since $P_{C_3}(k) = k(k-1)(k-2)$ and

$$(k-1)^3 - (k-1) = (k-1)\left[(k-1)^2 - 1\right] = (k-1)\left[k^2 - 2k\right] = k(k-1)(k-2).$$

2) Assume that $P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$ for some $n \geq 3$.

$$\begin{aligned}
\text{Then } P_{C_{n+1}}(k) &= P_{(C_{n+1})-e}(k) - P_{(C_{n+1})_e}(k) \\
&= P_{T_{n+1}}(k) - P_{C_n}(k) \\
&= k(k-1)^n - \left[(k-1)^n + (-1)^n(k-1)\right] \\
&= k(k-1)^n - (k-1)^n + (-1)^{n+1}(k-1) \\
&= (k-1)^n(k-1) + (-1)^{n+1}(k-1) \\
&= (k-1)^{n+1} + (-1)^{n+1}(k-1),
\end{aligned}$$

so the statement holds for $n+1$.