MATH 145

TEST 2 ANSWERS

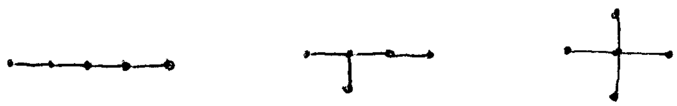
(1) A) even: 5 odd: 2 $\boxed{\text{OPEN}}$ (EXACTLY 2 ODD VERTICES)

B) even: 3 odd: 4 $\boxed{\text{NEITHER}}$ (MORE THAN 2 ODD VERTICES)

C) even: 8 odd: 5 $\boxed{\text{IMPOSSIBLE}}$ (NUMBER OF ODD VERTICES ISN'T EVEN)

D) even: 7 odd: 0 $\boxed{\text{CLOSED}}$ (ALL VERTICES ARE EVEN)

(2) A)



B) BY CAYLEY'S FORMULA, THERE ARE $n^{n-2} = 5^3 = \boxed{125}$ LABELED TREES OF ORDER 5.

(3) LET S BE THE SET OF ALL SUBSETS OF THE CLUB, OTHER THAN THE EMPTY SET AND THE ENTIRE CLUB, SO S HAS $2^8 - 2 = 254$ ELEMENTS.
THE POSSIBLE AGE SUM FOR AN ELEMENT OF S IS IN $\{1, \ldots, 245\}$ (SINCE $7(35) = 245$),
SO TWO ELEMENTS OF S HAVE THE SAME AGE SUM BY THE PH PRINCIPLE.
IF WE OMIT ANY COMMON MEMBERS, WE OBTAIN THE DESIRED GROUPS.

$\boxed{\text{EC}}$ 1) IF 2 PEOPLE, P_1 AND P_2, HAVE THE SAME AGE, THEN WE CAN TAKE THE TWO GROUPS TO BE $\{P_1\}$ AND $\{P_2\}$.

2) OTHERWISE, LET S BE THE SET OF ALL NONEMPTY SUBSETS OF THE CLUB, SO S HAS $2^8 - 1 = 255$ ELEMENTS.
THE AGE SUM OF ANY ELEMENT OF S IS IN $\{1, \ldots, 252\}$ SINCE $28 + 29 + \cdots + 35 = 252$,
SO BY THE PH PRINCIPLE THERE ARE 2 ELEMENTS OF S WITH THE SAME AGE SUM.
NOW WE CAN OMIT ANY COMMON MEMBERS TO GET 2 DISJOINT GROUPS.

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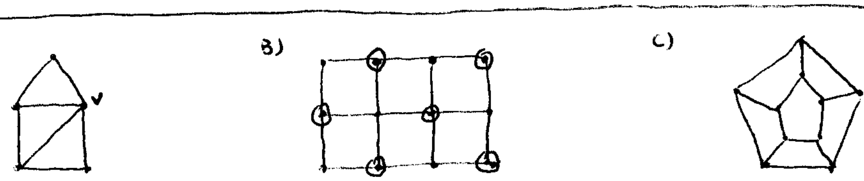
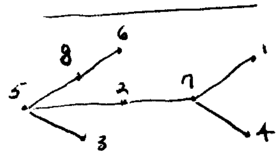
(4) A) B) C)



B) IS BIPARTITE AND A) AND C) ARE NOT, SINCE A) AND C) CONTAIN ODD CYCLES BUT B) HAS ONLY EVEN CYCLES.
(OR USE THAT A) IS NOT BIPARTITE BECAUSE VERTEX V IS ADJACENT TO ALL OTHERS, AND SHOW THAT B) IS BIPARTITE USING THE BIPARTITION SHOWN.)

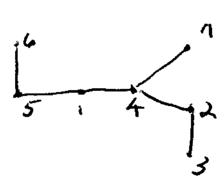
(5) SINCE $\deg(v) \le 5$ FOR EVERY VERTEX v, $\Delta \le 5$
SO $\chi(G) \le \Delta + 1 \le \boxed{6}$ (NOTICE THAT FOR $G = K_6$, THIS VALUE IS ATTAINED.)

(6) A)



$\boxed{757825}$ (SINCE THERE ARE 8 VERTICES, THE PRÜFER CODE HAS 6 DIGITS.)

B) $\dfrac{3\ 2\ 6\ 5\ 1}{2\ 4\ 5\ 1\ 4}$ (3 2) (2 4) (6 5) (5 1) (1 4) (4 7)



(SINCE THE PRÜFER CODE HAS 5 DIGITS, THERE ARE 7 VERTICES AND 6 EDGES.)

(7) $g(x) = (1+x+x^2+\cdots)(1+x+x^2+\cdots+x^7)(1+x^5+x^{10}+\cdots)(1+x^3+x^6+\cdots)(1+x+\cdots+x^4)(1+x+x^2)$

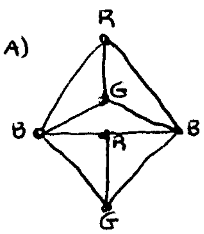
$$= \frac{1}{1-x}\cdot\frac{1-x^8}{1-x}\cdot\frac{1}{\cancel{1-x^5}}\cdot\frac{1}{\cancel{1-x^3}}\cdot\frac{\cancel{1-x^5}}{1-x}\cdot\frac{\cancel{1-x^3}}{1-x}$$

$$= \frac{1-x^8}{(1-x)^4} = (1-x^8)(1-x)^{-4} = (1-x^8)\sum_{m=0}^{\infty}\binom{m+3}{3}x^m$$

$$= \sum_{m=0}^{\infty}\binom{m+3}{3}x^m - \sum_{m=0}^{\infty}\binom{m+3}{3}x^{m+8} = \sum_{n=0}^{\infty}\binom{n+3}{3}x^n - \sum_{n=8}^{\infty}\binom{n-5}{3}x^n$$

SO $\boxed{a_n = \binom{n+3}{3} - \binom{n-5}{3} \text{ FOR } n \ge 8, \text{ AND } a_n = \binom{n+3}{3} \text{ FOR } 0 \le n < 8}$

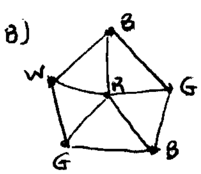
(8) A)



$\chi(G) \ge 3$, SINCE G CONTAINS K_3,

SO $\boxed{\chi(G) = 3}$ SINCE G HAS A 3-COLORING.

B)



SINCE C_5 REQUIRES 3 COLORS AND A 4TH COLOR IS REQUIRED FOR THE VERTEX IN THE CENTER, $\chi(G) \ge 4$;

SO $\boxed{\chi(G) = 4}$ SINCE G HAS A 4-COLORING.

(9) A) IF G IS A TREE OF ORDER n, THEN G HAS $n-1$ EDGES.

PF 1) IF $n=1$ THIS IS TRUE, SINCE G HAS 0 EDGES IF IT HAS ONLY ONE VERTEX.

2) ASSUME THAT THIS IS TRUE FOR n, AND LET G BE A TREE OF ORDER $n+1$.

IF v IS A LEAF OF G, THEN $G-v$ IS A TREE OF ORDER n SINCE

a) IT CONTAINS NO CYCLES AND

b) IT IS CONNECTED (BECAUSE NO PATH JOINING x AND y IN G WITH $x, y \ne v$ CAN USE THE DELETED EDGE).

THEREFORE $G-v$ HAS $n-1$ EDGES BY THE INDUCTION HYPOTHESIS,

SO G HAS n EDGES AND HENCE THE ASSERTION HOLDS FOR $n+1$.

REMARK I SHOULD HAVE SAID ANY TREE OF ORDER $n > 1$ HAS A LEAF.

B) IF G IS A GRAPH OF ORDER n WITH $n-1$ EDGES THAT HAS NO CYCLES, THEN G IS A TREE.

PF SINCE WE ARE ASSUMING THAT G HAS NO CYCLES, WE MUST SHOW THAT G IS CONNECTED:

LET $G_1, \ldots, G_k$ BE THE CONNECTED COMPONENTS OF G, WITH ORDERS $n_1, \ldots, n_k$, RESPECTIVELY.

THEN EACH G_i IS A TREE SINCE IT IS CONNECTED AND CONTAINS NO CYCLES,

SO G_i HAS $n_i - 1$ EDGES FOR EACH i BY PART A).

THEREFORE $n-1 = \sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - k = n - k$,

SO $k = 1$ AND THUS G IS CONNECTED.