MATH 145

SOLUTIONS - QUIZ 1

1. $a_n = 2a_{n-1} + 3a_{n-2}$, $a_0 = 9$, $a_1 = 7$

$r^2 = 2r + 3$, so $r^2 - 2r - 3 = 0$, $(r-3)(r+1) = 0$, $\underline{r=3}$ OR $\underline{r=-1}$

so $\underline{a_n = d(3^n) + e(-1)^n}$

$$\begin{aligned} a_0 &= d + e = 9 \\ a_1 &= 3d - e = 7 \\ \hline 4d &= 16 \end{aligned} \quad \text{(ADDING)}$$

$\underline{d = 4, \ e = 5}$

so $\boxed{a_n = 4(3^n) + 5(-1)^n}$

2. $$g_e(x) = \Big(\underbrace{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}_{(2,4)}\Big)^2 \Big(\underbrace{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}_{(5)}\Big)\Big(\underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}_{(1,3)}\Big)^2$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 (e^x - 1)(e^x)^2$$

$$= \frac{1}{4}(e^{2x} + 2 + e^{-2x})(e^x - 1)(e^{2x})$$

$$= \frac{1}{4}(e^{3x} + 2e^x + e^{-x} - e^{2x} - 2 - e^{-2x})(e^{2x})$$

$$= \underline{\frac{1}{4}(e^{5x} + 2e^{3x} + e^x - e^{4x} - 2e^{2x} - 1)}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left[5^n + 2\cdot 3^n + 1 - 4^n - 2\cdot 2^n\right]\cdot\frac{x^n}{n!},$$

so $\boxed{a_n = \frac{1}{4}\left[5^n + 2\cdot 3^n + 1 - 4^n - 2^{n+1}\right]}$ FOR $n \geq 1$, (WITH $a_0 = 0$).

3. $$g(x) = (1 + x + x^2 + \cdots)^2 (1 + x + x^2)(1 + x + x^2 + x^3)(1 + x^4 + x^8 + \cdots)(1 + x^3 + x^6 + \cdots)$$

$$= \left(\frac{1}{1-x}\right)^2 \cdot \frac{\cancel{1-x^3}}{1-x} \cdot \frac{\cancel{1-x^4}}{1-x} \cdot \frac{1}{\cancel{1-x^4}} \cdot \frac{1}{\cancel{1-x^3}} = \frac{1}{(1-x)^4}$$

$$= (1-x)^{-4} = \sum_{n=0}^{\infty}\binom{n+3}{3}x^n,$$ so $a_n = \boxed{\binom{n+3}{3}}$.

4. LET b_n, c_n, d_n BE THE NUMBER OF n-DIGIT STRINGS STARTING WITH 4, 5, 6 RESPECTIVELY.

1) $a_n = b_n + c_n + d_n$
2) $b_n = b_{n-1} + c_{n-1}$
3) $c_n = b_{n-1} + d_{n-1}$
4) $d_n = a_{n-1}$

so $a_n = (b_{n-1} + c_{n-1}) + (b_{n-1} + d_{n-1}) + a_{n-1}$
$= \underline{a_{n-1}} + b_{n-1} + a_{n-1} = 2a_{n-1} + b_{n-1}$
$= 2a_{n-1} + (b_{n-2} + c_{n-2}) = 2a_{n-1} + (a_{n-2} - d_{n-2})$
$= 2a_{n-1} + a_{n-2} - a_{n-3}$, so $\boxed{a_n = 2a_{n-1} + a_{n-2} - a_{n-3}}$

OR 1) IF 4 IS FIRST, THERE ARE $\underline{a_{n-1} - a_{n-2}}$ POSSIBILITIES (SINCE THERE ARE a_{n-2} POSSIBILITIES IF 6 IS THE 2ND LETTER)

$\underset{1}{\underline{4}}\ \underset{2}{\underline{\ }}\ \underset{3}{\underline{\ }}\ \underset{4}{\underline{\ }}\ \cdots\ \underset{n}{\underline{\ }}$

2) IF 6 IS FIRST, THERE ARE $\underline{a_{n-1}}$ POSSIBILITIES

$\underset{1}{\underline{6}}\ \underset{2}{\underline{\ }}\ \underset{3}{\underline{\ }}\ \cdots\ \underset{n}{\underline{\ }}$

(CHECK: $a_0 = 1$, $a_1 = 3$, $a_2 = 7$, AND $a_3 = 16$)

3) IF 5 IS FIRST, THERE ARE $\underbrace{(a_{n-2} - a_{n-3})}_{(4\text{ 2ND})} + \underbrace{a_{n-2}}_{(6\text{ 2ND})}$ POSSIBILITIES

$\underset{1}{\underline{5}}\ \underset{2}{\underline{4}}\ \underset{3}{\underline{\ }}\ \cdots\ \underset{n}{\underline{\ }}$ OR $\underset{1}{\underline{5}}\ \underset{2}{\underline{6}}\ \underset{3}{\underline{\ }}\ \cdots\ \underset{n}{\underline{\ }}$

THEREFORE $\boxed{a_n = 2a_{n-1} + a_{n-2} - a_{n-3}}$