MATH 145

# SOLUTIONS - QUIZ 2

(1) A) $K_5$ IS NOT A PLANAR GRAPH.

PF ASSUME INSTEAD THAT $K_5$ IS A PLANAR GRAPH.

THEN $n - e + f = 2$ GIVES $5 - 10 + f = 2$ SO $f = 7$,

AND $3f \le 2e$ IMPLIES $21 \le 20$.

THIS GIVES A CONTRADICTION, SO $K_5$ IS NOT PLANAR.

B) A CONNECTED PLANAR GRAPH HAS A VERTEX V WITH $\deg(v) \le 5$.

PF ASSUME INSTEAD THAT $\deg(v) \ge 6$ FOR EVERY VERTEX V.

(WE CAN ASSUME THAT $e \ge 2$, SINCE OTHERWISE $\deg(v) \le 1$ FOR EVERY VERTEX V.)

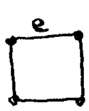
THEN $d_1 + \cdots + d_n = 2e \implies 2e \ge 6n \implies e \ge 3n$, AND

$n - e + f = 2 \implies 3n - 3e + 3f = 6 \implies 3n - 3e + 2e \ge 6 \implies e \le 3n - 6$.

THEREFORE $3n \le e \le 3n - 6$, WHICH GIVES A CONTRADICTION;

SO $\deg(v) \le 5$ FOR SOME VERTEX V.

REMARK BY #27 IN CH. 12, G MUST HAVE AT LEAST 2 VERTICES WITH $\deg(v) \le 5$ IF G HAS AT LEAST 2 VERTICES.

(2) 

IF e IS AN EDGE OF $C_4$,

$$P_G(k) = P_{G-e}(k) - P_{G_e}(k)$$

$$= P_{T_4}(k) - P_{K_3}(k)$$

$$= k(k-1)^3 - k(k-1)(k-2) = \boxed{k(k-1)(k^2 - 3k + 3)}$$

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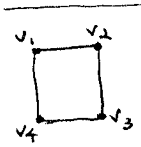
OR 

WE CONSIDER THE TWO CASES WHERE $v_1$ AND $v_3$ ARE COLORED THE SAME AND WHERE THEY ARE COLORED DIFFERENTLY:

$$P_G(k) = \underbrace{k(k-1)\cdot 1\cdot(k-1)}_{(v_1, v_3 \text{ SAME})} + \underbrace{k(k-1)(k-2)(k-2)}_{(v_1, v_3 \text{ DIFFERENT})} = \boxed{k(k-1)(k^2 - 3k + 3)}$$

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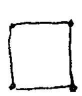
OR 

$$P_G(k) = \binom{k}{2} q_G(2) + \binom{k}{3} q_G(3) + \binom{k}{4} q_G(4)$$

$$= \frac{k(k-1)}{2}\cdot 2 + \frac{k(k-1)(k-2)}{6}\cdot 3\cdot 2\cdot 2 + \frac{k(k-1)(k-2)(k-3)}{24}\cdot 24$$

$$= k(k-1)\left[1 + 2(k-2) + (k-2)(k-3)\right] = \boxed{k(k-1)(k^2 - 3k + 3)}$$

(WHERE $q_G(3) = 3 \cdot 2 \cdot 2$ SINCE THERE ARE 3 CHOICES FOR THE COLOR THAT'S USED TWICE, 2 CHOICES FOR THE VERTICES THAT ARE COLORED THE SAME, AND 2 WAYS TO COLOR THE REMAINING 2 VERTICES)

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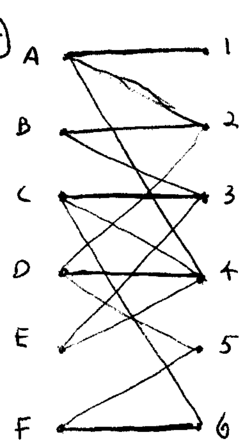
③ A) IF SOME VERTEX HAS DEGREE 0, THEN NO VERTEX HAS DEGREE 14, SO THE POSSIBLE DEGREES OF THE VERTICES ARE $\{0, 1, 2, \ldots, 13\}$. SINCE THERE ARE 15 VERTICES AND ONLY 14 POSSIBLE DEGREES, TWO VERTICES HAVE THE SAME DEGREE BY THE PH PRINCIPLE.

B) IF NO VERTEX HAS DEGREE 0, THEN THE POSSIBLE DEGREES ARE $\{1, \ldots, 14\}$. AS IN THE FIRST CASE, TWO VERTICES MUST HAVE THE SAME DEGREE BY THE PH PRINCIPLE.

THEREFORE IN EITHER CASE, TWO VERTICES HAVE THE SAME DEGREE.

REMARK THE SAME ARGUMENT APPLIES IF $G$ HAS ORDER $n \geq 2$, AND THIS PROBLEM IS EQUIVALENT TO #16 IN CH. 3.

④




M-AUGMENTING PATH: E4, 4D, D5

M′ = {A1, B2, C3, D5, E4, F6}

OR M-AUGMENTING PATH: E3, 3C, C4, 4D, D5

M′ = {A1, B2, C4, D5, E3, F6}

OR M-AUGMENTING PATH: E3, 3C, C6, 6F, F5

M′ = {A1, B2, C6, D4, E3, F5}

OR M-AUGMENTING PATH: E4, 4D, D2, 2B, B3, 3C, C6, 6F, F5

M′ = {A1, B3, C6, D2, E4, F5}