① 1) SEAT 20 SWEDES: $20!$ WAYS

2) CHOOSE 9 GAPS FOR THE GREEKS: $\binom{21}{9}$ WAYS

3) SEAT THE GREEKS IN THESE GAPS: $9!$ WAYS

ANSWER: $\boxed{20!\binom{21}{9}9!}$

$= \boxed{20!\,(21\cdot 20\cdot 19\cdots 13)}$

② $\boxed{\underbrace{\binom{60}{20}}_{\text{TOTAL}} - \underbrace{\binom{25}{20}}_{\text{0M}} - \underbrace{\binom{35}{1}\binom{25}{19}}_{\text{1M}} - \underbrace{\binom{35}{20}}_{\text{0W}}}$

③ A) LINE UP 35 DOTS, REPRESENTING THE INTEGERS NOT CHOSEN.

WE MUST CHOOSE 5 GAPS FOR THE DOTS REPRESENTING THE INTEGERS CHOSEN, WHICH CAN BE DONE $\boxed{\binom{36}{5}}$ WAYS (SINCE THERE ARE 36 GAPS).

OR

B) LINE UP 5 DOTS REPRESENTING THE INTEGERS CHOSEN, AND PUT ASIDE 4 BLOCKERS, THIS LEAVES 31 REMAINING DOTS, AND THERE ARE $\boxed{\binom{36}{5}}$ WAYS TO ARRANGE THE 5 CHOSEN DOTS AND THE 31 REMAINING DOTS.

(THEN INSERT THE 4 BLOCKERS IN THE 4 INNER GAPS.)

OR

C) | | | | | LET X_i BE THE NUMBER OF INTEGERS IN THE iTH GAP,

SO $\underline{X_1 + \cdots + X_6 = 35}$ WHERE $X_1, X_6 \geq 0$ AND $X_2, \ldots, X_5 \geq 1$.

IF WE LET $Y_1 = X_1$, $Y_6 = X_6$, AND $Y_i = X_i - 1$ FOR $2 \leq i \leq 5$,

WE GET $\underline{Y_1 + \cdots + Y_6 = 31}$ WITH $Y_i \geq 0$ FOR EACH i,

THIS HAS $\boxed{\binom{36}{5}}$ SOLUTIONS (31 DOTS, 5 DIVIDERS)

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④ $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, SO TAKING $\underline{x=-1}$ AND $\underline{y=r}$ GIVES

$\sum_{k=0}^{n} (-1)^k \binom{n}{k} r^{n-k} = \boxed{(-1+r)^n} = \boxed{(r-1)^n}$

⑤ $\boxed{\underbrace{\frac{11!}{3!\,2!\,2!\,2!}}_{\text{(TOTAL)}} - \underbrace{\frac{10!}{2!\,2!\,2!\,2!}}_{\text{(NUMBER STARTING WITH A)}}}$

OR THERE ARE 8 LETTERS OUT OF THE 11 WHICH ARE NOT AN A; SO SINCE EACH LETTER OUT OF THE 11 IS EQUALLY LIKELY TO BE FIRST, THERE ARE $\boxed{\frac{8}{11}\left(\frac{11!}{3!\,2!\,2!\,2!}\right)} = \boxed{\frac{8\cdot 10!}{3!\,2!\,2!\,2!}}$ SUCH ARRANGEMENTS

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⑥ 1) CHOOSE 3 INTEGERS TO BE FIXED: $\binom{8}{3}$ WAYS

2) DERANGE THE OTHER 5 INTEGERS: D_5 WAYS

ANSWER: $\underline{\binom{8}{3}\cdot D_5} = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}\cdot 44 = 56\cdot 44 = \boxed{2464}$

SINCE $D_5 = 5D_4 - 1 = 5\cdot 9 - 1 = \underline{44}$ (AND $D_4 = 4D_3 + 1 = 4\cdot 2 + 1 = 9$)

(OR) LET A_i BE THE SET OF PERMUTATIONS WHICH LEAVE i FIXED, FOR $1 \leq i \leq 8$,

(7) LET S BE THE SELECTIONS IF THERE WERE 20 OF EACH TYPE AVAILABLE, AND LET A_i BE THE SELECTIONS USING AT LEAST 6 BAGELS OF TYPE i, FOR $1 \le i \le 9$.

$$|\overline{A_1} \cap \cdots \cap \overline{A_9}| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \cdots$$

$$= \boxed{\binom{28}{8} - \binom{9}{1}\binom{22}{8} + \binom{9}{2}\binom{16}{8} - \binom{9}{3}\binom{10}{8}}$$

20 DOTS, 8 DIVIDERS; 14 DOTS, 8 DIVIDERS; 8 DOTS, 8 DIVIDERS; 2 DOTS, 8 DIVIDERS

(8) A) FIRST GIVE EACH STUDENT A BOOK.
THERE ARE 11 BOOKS REMAINING AND 4 DIVIDERS: $\boxed{\binom{15}{4}}$

OR

IF X_i IS THE NUMBER OF BOOKS RECEIVED BY STUDENT i,
$X_1 + \cdots + X_5 = 16$ WITH $X_i \ge 1$ FOR EACH i

WE HAVE 16 DOTS, AND WE MUST CHOOSE 4 OF THE 15 INNER GAPS INTO WHICH WE CAN PUT THE DIVIDERS: $\boxed{\binom{15}{4}}$

B) LET S BE ALL DISTRIBUTIONS, AND
LET A_i BE THE SET OF DISTRIBUTIONS WHERE STUDENT i DOES NOT GET A BOOK, $1 \le i \le 5$

$$|\overline{A_1} \cap \cdots \cap \overline{A_5}| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \cdots$$

$$= \boxed{5^{16} - \binom{5}{1}4^{16} + \binom{5}{2}3^{16} - \binom{5}{3}2^{16} + \binom{5}{4}}$$

(9) LET S BE ALL SEATINGS OF THE 8 PEOPLE, AND
LET A_i BE THE SEATINGS WITH COUPLE i TOGETHER FOR $1 \le i \le 4$.

$$|\overline{A_1} \cap \cdots \cap \overline{A_4}| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + |A_1 \cap \cdots \cap A_4|$$

$$= \boxed{7! - \binom{4}{1}6! \cdot 2 + \binom{4}{2}5! \cdot 2^2 - \binom{4}{3}4! \cdot 2^3 + \binom{4}{4}3! \cdot 2^4}$$

(IF THERE ARE K COUPLES TOGETHER, THEN THERE ARE $\underline{(7-K)!}$ WAYS TO ARRANGE THE $8-K$ "UNITS" IN A CIRCLE; AND SINCE THERE ARE 2 WAYS TO ORDER EACH COUPLE, THERE ARE $\underline{2^K}$ WAYS TO ORDER THE K COUPLES.)

(10) THERE ARE 5 CASES TO CONSIDER:

1) NO TIES: $5!$ OUTCOMES
2) 1 PAIR TIE: $\binom{5}{2} \cdot 4!$ OUTCOMES
3) 2 PAIRS TIE: $\binom{5}{4} \cdot 3 \cdot 3!$ OUTCOMES
4) 3 PEOPLE TIE: $\binom{5}{3} \cdot 3!$ OUTCOMES
5) 1 PAIR AND 1 TRIPLE TIE: $\binom{5}{2} \cdot 2!$ OUTCOMES

3) a) SELECT 4 PEOPLE: $\binom{5}{4}$ WAYS
b) PAIR THEM UP: $3 \cdot 1$ WAYS
c) ORDER THE 3 UNITS: $3!$ WAYS

OR a) SELECT THE LEADING PAIR: $\binom{5}{2}$ WAYS
b) SELECT THE OTHER PAIR: $\binom{3}{2}$ WAYS
c) SELECT THE PLACE FOR THE LAST PERSON: 3 WAYS

ANSWER: $\boxed{5! + \binom{5}{2} \cdot 4! + \binom{5}{4} \cdot 3 \cdot 3! + \binom{5}{3} \cdot 3! + \binom{5}{2} \cdot 2!} = \boxed{530}$

(11) LET c_n, d_n, e_n, f_n BE THE NUMBER OF CODEWORDS WITH n LETTERS WHICH START WITH C, D, E, F, RESPECTIVELY.

THEN 1) $a_n = c_n + d_n + e_n + f_n$

2) $e_n = a_{n-1}$

$f_n = a_{n-1}$

$c_n = c_{n-1} + e_{n-1} + f_{n-1}$

$d_n = d_{n-1} + e_{n-1} + f_{n-1}$

THEREFORE $a_n = 2a_{n-1} + (c_{n-1} + d_{n-1} + e_{n-1} + f_{n-1}) + e_{n-1} + f_{n-1}$

$= 2a_{n-1} + a_{n-1} + a_{n-2} + a_{n-2}$,

SO $\boxed{a_n = 3a_{n-1} + 2a_{n-2}}$

(CHECK: $a_0 = 1$, $a_1 = 4$, $a_2 = 14$, $a_3 = 50 \leftarrow 4^3 - 4 \cdot 4 + 2$, SINCE DCD AND CDC GET SUBTRACTED TWICE)