MATH 145 — CH. 6 – GENERALIZED INCL.-EXCL. PRINCIPLE

WE CAN GENERALIZE THE INCLUSION-EXCLUSION PRINCIPLE AS FOLLOWS:

**TH** LET $A_1, \dots, A_n$ BE SUBSETS OF A FINITE SET, AND

LET $T_k = \sum |A_{i_1} \cap \dots \cap A_{i_k}|$ FOR $1 \le k \le n$,

WHERE THE SUM IS TAKEN OVER ALL $k$-COMBINATIONS OF $\{1, \dots, n\}$.

① IF $N_m$ IS THE NUMBER OF ELEMENTS IN AT LEAST $m$ OF THE SETS $A_1, \dots, A_n$,

$$\boxed{N_m = T_m - \binom{m}{m-1} T_{m+1} + \binom{m+1}{m-1} T_{m+2} - \binom{m+2}{m-1} T_{m+3} + \dots + (-1)^{n-m} \binom{n-1}{m-1} T_n}.$$

② IF $E_m$ IS THE NUMBER OF ELEMENTS IN EXACTLY $m$ OF THE SETS $A_1, \dots, A_n$,

$$\boxed{E_m = T_m - \binom{m+1}{m} T_{m+1} + \binom{m+2}{m} T_{m+2} - \binom{m+3}{m} T_{m+3} + \dots + (-1)^{n-m} \binom{n}{m} T_n}.$$

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REMARKS

A) PART ① WITH $m = 1$ IS JUST THE INCLUSION-EXCLUSION PRINCIPLE!

$$N_1 = T_1 - T_2 + T_3 - \dots + (-1)^{n-1} T_n.$$

THE PROOF OF PART ① IS A COUNTING ARGUMENT SIMILAR TO THE PROOF OF THE INCL.-EXCL. PRINCIPLE.

B) PART ② FOLLOWS FROM PART ①, USING $E_m = N_m - N_{m+1}$

AND PASCAL'S FORMULA: $\binom{k}{m} = \binom{k-1}{m} + \binom{k-1}{m-1}$.

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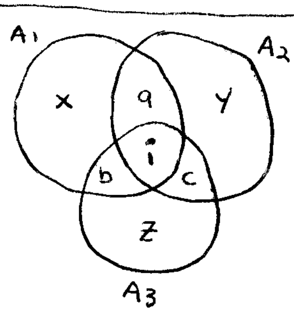
Ex




1) $N_2 = T_2 - 2T_3 = \sum |A_i \cap A_j| - 2|A_1 \cap A_2 \cap A_3|$

$= \big[(a+i) + (b+i) + (c+i)\big] - 2i = a + b + c + i$.

2) $E_1 = T_1 - 2T_2 + 3T_3$

$= \sum |A_i| - 2\sum |A_i \cap A_j| + 3|A_1 \cap A_2 \cap A_3|$

$= \big[(x+a+b+i) + (y+a+c+i) + (z+b+c+i)\big] -$

$2\big[(a+i) + (b+i) + (c+i)\big] + 3i = x + y + z$

3) $E_2 = T_2 - 3T_3 = \sum |A_i \cap A_j| - 3|A_1 \cap A_2 \cap A_3|$

$= \big[(a+i) + (b+i) + (c+i)\big] - 3i = a + b + c$